Real Analysis, Quantitative Topology, and Geometric Complexity

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1 Mappings and distortion

A very basic mechanism for controlling geometric complexity is to limit the way that distances can be distorted by a mapping.

If distances are distorted by only a small amount, then one might think of the mapping as being approximately "flat". Let us look more closely at this, and see what actually happens.

Let δ be a small positive number, and let f be a mapping from the Euclidean plane $\mathbf{R}^2$ to itself. Given two points $x, y \in \mathbf{R}^2$, let $|x - y|$ denote the usual Euclidean distance between them. We shall assume that

$$(1+\delta)^{-1} |x-y| \le |f(x) - f(y)| \le (1+\delta) |x-y| \tag{1.1}$$

for all $x, y \in \mathbf{R}^2$. This says exactly that f does not ever shrink or expand distances by more than a factor of $1 + \delta$.

What does this really mean about the behavior of f? A first point is that if δ were equal to 0, so that f does not distort distances at all, then f would have to be a "rigid" mapping. This means that f could be expressed as

$$f(x) = A(x) + b, \tag{1.2}$$

where b is an element of $\mathbf{R}^2$ and A is a linear mapping on $\mathbf{R}^2$ which is either a rotation or a combination of a rotation and a reflection. This is well known, and it is not hard to prove. For instance, it is not hard to show that the assumption that f preserve distances implies that f takes lines to lines, and that it preserve angles, and from there it is not hard to see that f must be of the form (1.2) as above.

If δ is not equal to zero, then one would like to say that f is approximately equal to a rigid mapping when δ is small enough. Here is a precise statement. Let D be a (closed) disk of radius r in the plane. This means that there is a point $w \in \mathbf{R}^2$ such that

$$D = \{x \in \mathbf{R}^2 : |x - w| \le r\}. \tag{1.3}$$

Then there is a rigid mapping $T : \mathbf{R}^2 \to \mathbf{R}^2$, depending on D and f, such that

$$r^{-1} \sup_{x \in D} |f(x) - T(x)| \le \text{small}(\delta), \tag{1.4}$$

where small(δ) depends only on δ, and not on D or f, and has the property that

$$\text{small}(\delta) \to 0 \quad \text{as} \quad \delta \to 0. \tag{1.5}$$

There are a number of ways to look at this. One can give direct constructive arguments, through basic geometric considerations or computations. In particular, one can derive explicit bounds for small(δ) in terms of δ. Results of this kind are given in [Joh]. There are also abstract and inexplicit methods, in which one argues by contradiction using compactness and the Arzela–Ascoli theorem. (In some related but different contexts, this can be fairly easy or manageable, while explicit arguments and estimates are less clear.)

The presence of the factor of r^{-1} on the left side of (1.4) may not make sense at first glance, but it is absolutely on target, and indispensable. It reflects the natural *scaling* of the problem, and converts the left-hand side of (1.4) into a dimensionless quantity, just as δ is dimensionless. One can view this in terms of the natural *invariances* of the problem. Nothing changes here if we compose f (on either side) with a translation, rotation, or reflection, and the same is true if we make simultaneous dilations on both the domain and the range of equal amounts. In other words, if a is any positive number, and if we define $f_a : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$f_a(x) = a^{-1} f(ax), \tag{1.6}$$

then f_a satisfies (1.1) exactly when f does. The approximation condition (1.4) is formulated in such a way as to respect the same kind of invariances as (1.1) does, and the factor of r^{-1} accounts for the dilation-invariance.

This kind of approximation by rigid mappings is pretty good, but can we do better? Is it possible that the approximation works at the level of the *derivatives* of the mappings, rather than just the mappings themselves?

Here is another way to think about this, more directly in terms of distance geometry. Let us consider a simple mechanism by which mappings that satisfy (1.1) can be produced, and ask whether this mechanism gives everything. Fix a nonnegative number k, and call a mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ is *k-Lipschitz* if

$$|g(x) - g(y)| \le k\,|x - y| \tag{1.7}$$

for all $x, y \in \mathbf{R}^2$. This condition is roughly equivalent to saying that the differential of g has norm less than or equal to k everywhere. Specifically, if g is differentiable at every point in $\mathbf{R}^2$, and if the norm of its differential is bounded by k everywhere, then (1.7) holds, and this can be derived from the mean value theorem. The converse is not quite true, however, because Lipschitz mappings need not be differentiable everywhere. They

are differentiable *almost everywhere*, in the sense of Lebesgue measure. (See [Fed, Ste1, Sem12].) To get a proper equivalence one can consider derivatives in the sense of distributions.

If $f = S + g$, where S is a rigid mapping and g is k-Lipschitz, and if $k \le 1/2$ (say), then f satisfies (1.1) with $\delta = 2k$. (More precisely, one can take $\delta = (1-k)^{-1} - 1$.) This is not hard to check. When k is small, this is a much stronger kind of approximation of f by rigid mappings than (1.4) is. In particular, it implies that the differential of f is uniformly close to the differential of S.

To what extent can one go in the opposite direction, and say that if f satisfies (1.1) with δ small, then f can be approximated by rigid mappings in this stronger sense? Let us begin by looking at what happens with the differential of f at individual points. Let x be some point in $\mathbf{R}^2$, and assume that the differential df_x of f at x exists. Thus df_x is a linear mapping from $\mathbf{R}^2$ to itself, and

$$f(x) + df_x(y - x) \tag{1.8}$$

provides a good approximation to $f(y)$ for y near x, in the sense that

$$|f(y) - \{f(x) + df_x(y - x)\}| = o(|y - x|). \tag{1.9}$$

One can also think of the differential as the map obtained from f by "blowing up" at x. This corresponds to the formula

$$df_x(v) = \lim_{t \to 0} t^{-1}(f(x + tv) - f(x)), \tag{1.10}$$

with t taken from positive real numbers.

It is not hard to check that df_x, as a mapping on $\mathbf{R}^2$ (with x fixed), automatically satisfies (1.1) when f does. Because the differential is already linear, standard arguments from linear algebra imply that it is close to a rotation or to the composition of a rotation and a reflection when δ is small, and with easy and explicit estimates for the degree of approximation.

This might sound pretty good, but it is actually much weaker than something like a representation of f as $S + g$, where S is a rigid mapping and g is k-Lipschitz with a reasonably-small value of k. If there is a representation of this type, then it means that the differential df_x of f is always close to the differential of S, which is *constant*, i.e., independent of x. The simple method of the preceding paragraph implies that df_x is always close to being a rotation or a rotation composed with a reflection, but a priori the choice

of such a linear mapping could depend on x in a strong way. That is very different from saying that there is a single linear mapping that works for every x.

Here is an example which shows how this sort of phenomenon can happen. (See also [Joh].) Let us work in polar coordinates, so that a point z in $\mathbf{R}^2$ is represented by a radius $r \geq 0$ and an angle θ. We define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by saying that if x is described by the polar coordinates (r, θ), then

$$f(x) \text{ has polar coordinates } (r, \theta + \epsilon \log r). \tag{1.11}$$

Here ϵ is a small positive number that we get to choose. Of course f should also take the origin to itself, despite the fact that the formula for the angle degenerates there.

Thus f maps each circle centered at the origin to itself, and on each such circle f acts by a rotation. We do not use a single rotation for the whole plane, but instead let the rotation depend logarithmically on the radius, as above. This has the effect of transforming every line through the origin into a logarithmic spiral. This spiral is very "flat" when ϵ is small, but nonetheless it does wrap around the origin infinitely often in every neighborhood of the origin.

It is not hard to verify that this construction leads to a mapping f that satisfies (1.1), with a δ that goes to 0 when ϵ does, and at an easily computable (linear) rate. This gives an example of a mapping that cannot be represented as $S + g$ with S rigid and g k-Lipschitz for a fairly small value of k (namely, $k < 1$). For if f did admit such a representation, it would not be able to transform lines into curves that spiral around a fixed point infinitely often; instead it would take a line L to a curve Γ which can be realized as the graph of a function over the line $S(L)$. The spirals that we get can never be realized as a graph of a function over any line. This is not hard to check.

This spiralling is not incompatible with the kind of approximation by rigid mappings in (1.4). Let us consider the case where D is a disk centered at the origin, which is the worst-case scenario anyway. One might think that (1.4) fails when we get too close to the origin (as compared to the radius of D), but this is not the case. Let T be the rotation on $\mathbf{R}^2$ that agrees with f on the boundary of D. If ϵ is small (which is necessary in order for the δ to be small in (1.1)), then T provides a good approximation to f on D in the sense of (1.4). In fact, T provides a good approximation to f at the level of their derivatives too on most of D, i.e., on the complement of a

small neighborhood of the origin. The approximation of derivatives breaks down near the origin, but the approximation of values does not, as in (1.4), because f and T both take points near the origin to points near the origin.

This example suggests another kind of approximation by rigid mappings that might be possible. Given a disk D of radius r and a mapping f that satisfies (1.1), one would like to have a rigid mapping T on $\mathbf{R}^2$ so that (1.4) holds, and also so that

$$\frac{1}{\pi r^2} \int_D \|df_x - dT\| \, dx \le \text{small}'(\delta), \tag{1.12}$$

where $\text{small}'(\delta)$ is, as before, a positive quantity which depends only on δ (and not on f or D) and which tends to 0 when δ tends to 0. Here dx refers to the ordinary 2-dimensional integration against area on $\mathbf{R}^2$, and we think of $df_x - dT$ as a matrix-valued function of x, with $\|df_x - dT\|$ denoting its norm (in any reasonable sense).

In other words, instead of asking that the differential of f be approximated *uniformly* by the differential of a rigid mapping, which is not true in general, one can ask only that the differential of f be approximated by the differential of T *on average*.

This does work, and in fact one can say more. Consider the expression

$$P(\lambda) = \text{Probability}(\{x \in D : \|df_x - dT\| \ge \text{small}'(\delta) \cdot \lambda\}), \tag{1.13}$$

where λ is a positive real number. Here "probability" means Lebesgue measure divided by πr^2, which is the total measure of the disk D. The inequality (1.12) implies that

$$P(\lambda) \le \frac{1}{\lambda} \tag{1.14}$$

for all $\lambda > 0$. It turns out that there is actually a universal bound for $P(\lambda)$ with exponential decay for $\lambda \ge 1$. This was proved by John [Joh] (with concrete estimates).

Notice that uniform approximation of the differential of f by the differential of T would correspond to a statement like

$$P(\lambda) = 0 \tag{1.15}$$

for all λ larger than some fixed (universal) constant. John's result of exponential decay is about the next best thing.

As a technical point, let us mention that one can get exponential decay conditions concerning the way that $\|df_x - dT\|$ should be small most of the time in a kind of trivial manner, with constants that are not very good (at all), using the linear decay conditions with good constants, together with the fact that df is bounded, so that $\|df_x - dT\|$ is bounded. In the exponential decay result mentioned above, the situation is quite different, and one keeps constants like those from the linear decay condition. This comes out clearly in the proof, and we shall see more about it later.

This type of exponential decay occurs in a simple way in the example above, in (1.11). (This also comes up in [Joh].) One can obtain this from the presence of $\epsilon \log r$ in the angle coordinate in the image. The use of the logarithm here is not accidental, but fits exactly with the requirements on the mapping. For instance, if one differentiates $\log r$ in ordinary Cartesian coordinates, then one gets a quantity of size $1/r$, and this is balanced by the r in the first part of the polar coordinates in (1.11), to give a result which is bounded.

It may be a bit surprising, or disappointing, that uniform approximation to the differential of f does not work here. After all, we did have "uniform" (or "supremum") bounds in the hypothesis (1.1), and so one might hope to have the same kind of bounds in the conclusion. This type of failure of supremum bounds is quite common, and in much the same manner as in the present case. We shall return to this in Section 2.

How might one prove (1.12), or the exponential decay bounds for $P(\lambda)$? Let us start with a slightly simpler situation. Imagine that we have a rectifiable curve γ in the plane whose total length is only slightly larger than the distance between its two endpoints. If the length of γ were *equal* to the distance between the endpoints, then γ would have to be a straight line segment, and nothing more. If the length is slightly larger, then γ has to stay close to the line segment that joins its endpoints. In analogy with (1.12), we would like to say that the tangents to γ are nearly parallel, on average, to the line that passes through the endpoints of γ.

In order to analyze this further, let $z(t)$, $t \in \mathbf{R}$, $a \le t \le b$, be a parameterization of γ by arclength. This means that $z(t)$ should be 1-Lipschitz, so that

$$|z(s) - z(t)| \le |s - t| \tag{1.16}$$

for all $s, t \in [a, b]$, and that $|z'(t)| = 1$ for almost all t, where $z'(t)$ denotes

the derivative of $z(t)$. Set

$$\zeta = \frac{z(b) - z(a)}{b - a} = \frac{1}{b - a} \int_a^b z'(t)\, dt. \tag{1.17}$$

Let us compute

$$\frac{1}{b - a} \int_a^b |z'(s) - \zeta|^2\, ds, \tag{1.18}$$

which controls the average oscillation of $z'(s)$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbf{R}^2$, so that

$$\begin{aligned} |x - y|^2 = \langle x - y, x - y \rangle &= \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle \\ &= |x|^2 - 2\langle x, y \rangle + |y|^2 \end{aligned} \tag{1.19}$$

for all $x, y \in \mathbf{R}^2$. Applying this with $x = z'(s)$, $y = \zeta$, we get that

$$\frac{1}{b - a} \int_a^b |z'(s) - \zeta|^2\, ds = 1 - 2\frac{1}{b - a} \int_a^b \langle z'(s), \zeta \rangle\, ds + |\zeta|^2, \tag{1.20}$$

since $|z'(s)| = 1$ a.e., and ζ does not depend on s. The middle term on the right side reduces to

$$2\langle \zeta, \zeta \rangle, \tag{1.21}$$

because of (1.17). Thus (1.20) yields

$$\frac{1}{b - a} \int_a^b |z'(s) - \zeta|^2\, ds = 1 - 2|\zeta|^2 + |\zeta|^2 = 1 - |\zeta|^2. \tag{1.22}$$

On the other hand, $|z(b) - z(a)|$ is the same as the distance between the endpoints of γ, and $b - a$ is the same as the length of γ, since $z(t)$ is the parameterization of γ by arclength. Thus $|\zeta|$ is exactly the ratio of the distance between the endpoints of γ to the length of γ, by (1.17), and $1 - |\zeta|^2$ is a dimensionless quantity which is small exactly when the length of γ and the distance between its endpoints are close to each other (proportionately). In this case (1.22) provides precise information about the way that $z'(s)$ is approximately a constant on average. (These computations follow ones in [CoiMe2].)

One can use these results for curves for looking at mappings from $\mathbf{R}^2$ (or $\mathbf{R}^n$) to itself, by considering images of segments under the mappings. This does not seem to give the proper bounds in (1.12), in terms of dependence on δ, though. In this regard, see John's paper [Joh]. (Compare also with

Appendix A.) Note that for curves by themselves, the computations above are quite sharp, as indicated by the equality in (1.22). See also [CoiMe2].

The exponential decay of $P(\lambda)$ requires more work. A basic point is that exponential decay bounds can be derived in a very general way once one knows (1.12) for *all* disks D in the plane. This is a famous result of John and Nirenberg [JohN], which will be discussed further in Section 2. In the present situation, having estimates like (1.12) for all disks D (and with uniform bounds) is quite natural, and is essentially automatic, because of the invariances of the condition (1.1) under translations and dilations. In other words, once one has an estimate like (1.12) for some fixed disk D and all mappings f which satisfy (1.1), one can conclude that the same estimate works for all disks D, because of invariance under translations and dilations.

2 The mathematics of good behavior much of the time, and the BMO frame of mind

Let us start anew for the moment, and consider the following question in analysis. Let h be a real-valued function on $\mathbf{R}^2$. Let Δ denote the Laplace operator, given by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{2.1}$$

where x_1, x_2 are the standard coordinates on $\mathbf{R}^2$. To what extent does the behavior of Δh control the behavior of the other second derivatives of h?

Of course it is easy to make examples where Δh vanishes at a point but the other second derivatives do not vanish at the same point. Let us instead look for ways in which the overall behavior of Δh can control the overall behavior of the other second derivatives.

Here is a basic example of such a result. Let us assume (for simplicity) that h is smooth and that it has compact support, and let us write ∂_1 and ∂_2 for $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively. Then

$$\int_{\mathbf{R}^2} |\partial_1 \partial_2 h(x)|^2 \, dx \le \int_{\mathbf{R}^2} |\Delta h(x)|^2 \, dx. \tag{2.2}$$

This is a well-known fact, and it can be derived as follows. We begin with the identity

$$\int_{\mathbf{R}^2} \partial_1 \partial_2 h(x) \, \partial_1 \partial_2 h(x) \, dx = \int_{\mathbf{R}^2} \partial_1^2 h(x) \, \partial_2^2 h(x) \, dx, \tag{2.3}$$

which uses two integrations by parts. On the other hand,

$$\int_{\mathbf{R}^2} |\Delta h(x)|^2 \, dx = \int_{\mathbf{R}^2} (\partial_1^2 h(x) + \partial_2^2 h(x))^2 \, dx \tag{2.4}$$
$$= \int_{\mathbf{R}^2} (\partial_1^2 h(x))^2 + 2 \, \partial_1^2 h(x) \, \partial_2^2 h(x) + (\partial_2^2 h(x))^2 \, dx.$$

Combining this with (2.3) we get that

$$\int_{\mathbf{R}^2} |\Delta h(x)|^2 \, dx - 2 \int_{\mathbf{R}^2} |\partial_1 \partial_2 h(x)|^2 \, dx \tag{2.5}$$
$$= \int_{\mathbf{R}^2} (\partial_1^2 h(x))^2 + (\partial_2^2 h(x))^2 \, dx.$$

This implies (2.2), and with an extra factor of 2 on the left-hand side, because the right side of (2.5) is nonnegative. (One can improve this to get a factor of 4 on the left side of (2.2), using the right-hand side of (2.5).)

In short, the L^2 norm of Δh always bounds the L^2 norm of $\partial_1 \partial_2 h$. There are similar bounds for L^p norms when $1 < p < \infty$. Specifically, for each p in $(1, \infty)$, there is a constant $C(p)$ such that

$$\int_{\mathbf{R}^2} |\partial_1 \partial_2 h(x)|^p \, dx \le C(p) \int_{\mathbf{R}^2} |\Delta h(x)|^p \, dx \tag{2.6}$$

whenever h is a smooth function with compact support. This is a typical example of a "Calderón–Zygmund inequality", as in [Ste1]. Such inequalities do *not* work for $p = 1$ or ∞, and the $p = \infty$ case is like the question of supremum estimates in Section 1. Note that the $p = 1$ and $p = \infty$ cases are closely connected to each other, because of duality (of spaces and operators); the operators Δ and $\partial_1 \partial_2$ here are equal to their own transposes, with respect to the standard bilinear form on functions on $\mathbf{R}^2$ (defined by taking the integral of the product of two given functions). In a modestly different direction, there are classical results which give bounds in terms of the norm for Hölder continuous (or Lipschitz) functions of order α, for every $\alpha \in (0, 1)$, instead of the L^p norm. To be explicit, given α, this norm for a function g on $\mathbf{R}^2$ can be described as the smallest constant A such that

$$|g(x) - g(y)| \le A \, |x - y|^\alpha \tag{2.7}$$

for all $x, y \in \mathbf{R}^2$. One can view this as a $p = \infty$ situation, like the L^∞ norm for g, but with a positive order α of smoothness, unlike L^∞. There is

a variety of other norms and spaces which one can consider, and for which there are results about estimates along the lines of (2.6), but for the norm in question instead of the L^p norm.

The $p = \infty$ version of (2.6) would say that there is a constant C such that

$$\sup_{x \in \mathbf{R}^2} |\partial_1 \partial_2 h(x)| \le C \sup_{x \in \mathbf{R}^2} |\Delta h(x)| \tag{2.8}$$

whenever h is smooth and has compact support. In order to see that this is not the case, consider the function $h(x)$ given by

$$h(x) = x_1 x_2 \log(x_1^2 + x_2^2), \tag{2.9}$$

$x = (x_1, x_2)$. It is not hard to compute Δh and $\partial_1 \partial_2 h$ explicitly, and to see that Δh is bounded while $\partial_1 \partial_2 h$ is not. Indeed,

$$\partial_1 \partial_2 h(x) = \log(x_1^2 + x_2^2) + \text{bounded terms}, \tag{2.10}$$

while the logarithm does not survive in Δh, because $\Delta(x_1 x_2) \equiv 0$.

This choice of h is neither smooth nor compactly supported, but these defects can be corrected easily. For smoothness we can consider instead

$$h_\epsilon(x) = x_1 x_2 \log(x_1^2 + x_2^2 + \epsilon), \tag{2.11}$$

where $\epsilon > 0$, and then look at what happens as $\epsilon \to 0$. To make the support compact we can simply multiply by a fixed cut-off function that does not vanish at the origin. With these modifications we still get a singularity at the origin as $\epsilon \to 0$, and we see that (2.8) cannot be true (with a fixed constant C that does not depend on h).

This is exactly analogous to what happened in Section 1, i.e., with a uniform bound going in but not coming out. Instead of a uniform bound for the output, we also have a substitute in terms of "mean oscillation", just as before. To be precise, let D be any disk in $\mathbf{R}^2$ of radius r, and consider the quantity

$$\frac{1}{\pi r^2} \int_D |\partial_1 \partial_2 h(x) - \text{Average}_D(\partial_1 \partial_2 h)| \, dx, \tag{2.12}$$

where "$\text{Average}_D \partial_1 \partial_2 h$" is the average of $\partial_1 \partial_2 h$ over the disk D, i.e.,

$$\text{Average}_D(\partial_1 \partial_2 h) = \frac{1}{\pi r^2} \int_D \partial_1 \partial_2 h(u) \, du. \tag{2.13}$$

Instead of (2.8), it is true that there is a constant $C > 0$ so that

$$\frac{1}{\pi r^2} \int_D |\partial_1 \partial_2 h(x) - \text{Average}_D(\partial_1 \partial_2 h)| \, dx \le C \sup_{x \in \mathbf{R}^2} |\Delta h(x)| \tag{2.14}$$

for every disk D in $\mathbf{R}^2$ of radius r and every smooth function h with compact support. This is not too hard to prove; roughly speaking, the point is to "localize" the L^2 estimate that we had before. (More general results of this nature are discussed in [GarcR, Jou, Ste2].)

Let us formalize this estimate by defining a new space of functions, namely the space BMO of functions of *bounded mean oscillation*, introduced by John and Nirenberg in [JohN]. A locally-integrable function g on $\mathbf{R}^2$ is said to lie in BMO if there is a nonnegative number k such that

$$\frac{1}{\pi r^2} \int_D |g(x) - \text{Average}_D(g)| \, dx \le k \tag{2.15}$$

for every disk D in $\mathbf{R}^2$ of radius r. In this case we set

$$\|g\|_* = \sup_D \frac{1}{\pi r^2} \int_D |g(x) - \text{Average}_D(g)| \, dx, \tag{2.16}$$

with the supremum taken over all disks D in $\mathbf{R}^2$. This is the same as saying that $\|g\|_*$ is the *smallest* number k that satisfies (2.15). One refers to $\|g\|_*$ as the "BMO norm of g", but notice that $\|g\|_* = 0$ when g is equal to a constant almost everywhere. (The converse is also true.)

This definition may look a little crazy, but it works quite well in practice. Let us reformulate (2.14) by saying that there is a constant C so that

$$\|\partial_1 \partial_2 h\|_* \le C \, \|\Delta h\|_\infty, \tag{2.17}$$

where $\|\phi\|_\infty$ denotes the L^∞ norm of a given function ϕ. In other words, although the L^∞ norm of $\partial_1 \partial_2 h$ is not controlled (for all h) by the L^∞ norm of Δh, the BMO norm of $\partial_1 \partial_2 h$ is controlled by the L^∞ norm of Δh.

Similarly, one of the main points in Section 1 can be reformulated as saying that if a mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ distorts distances by only a small amount, as in (1.1), then the BMO norm $\|df\|_*$ of the differential of f is small (and with precise estimates being available).

In Section 1 we mentioned a stronger estimate with exponential decay in the measure of certain "bad" sets. This works for all BMO functions,

and can be given as follows. Suppose that g is a BMO function on $\mathbf{R}^2$ with $\|g\|_* \le 1$, and let D be a disk in $\mathbf{R}^2$ with radius r. As in (1.13), consider the "distribution function" $P(\lambda)$ defined by

$$P(\lambda) = \text{Probability}(\{x \in D : |g(x) - \text{Average}_D(g)| \ge \lambda\}), \tag{2.18}$$

where "Probability" means Lebesgue measure divided by the area πr^2 of D. Under these conditions, there is a universal bound for $P(\lambda)$ with exponential decay, i.e., an inequality of the form

$$P(\lambda) \le B^{-\lambda} \qquad \text{for } \lambda \ge 1, \tag{2.19}$$

where B is a positive number greater than 1, and B does not depend on g or D. This is a theorem of John and Nirenberg [JohN].

Although we have restricted ourselves to $\mathbf{R}^2$ here for simplicity, everything goes over in a natural way to Euclidean spaces of arbitrary dimension. In fact, there is a much more general framework of "spaces of homogeneous type" in which basic properties of BMO (and other aspects of real-variable harmonic analysis) carry over. See [CoiW1, CoiW2], and compare also with [GarcR, Ste2]. This framework includes certain Carnot spaces that arise in several complex variables, like the unit sphere in $\mathbf{C}^n$ with the appropriate (noneuclidean) metric.

The exponential decay bound in (2.19) helps to make precise the idea that BMO functions are very close to being bounded (which would correspond to having $P(\lambda) = 0$ for all sufficiently large λ). The exponential rate of decay implies that BMO functions lie in L^p locally for all finite p, but it is quite a bit stronger than that.

A basic example of a BMO function is $\log |x|$. This is not hard to check, and it shows that exponential decay in (2.19) is sharp, i.e., one does not have superexponential decay in general. This example also fits with (2.10), and with the "rotational" part of the differential of the mapping f in (1.11).

In general, BMO functions can be much more complicated than the logarithm. Roughly speaking, the total "size" of the unboundedness is no worse than for the logarithm, as in (2.19), but the arrangement of the singularities can be more intricate, just as one can make much more complex singular examples than in (2.9) and (1.11). There are a lot of tools available in harmonic analysis for understanding exactly how BMO functions behave. (See [GarcR, Garn, Jou, Ste2], for instance.)

BMO functions show up all over the place. One can reformulate the basic scenario in this section with the Laplacian and $\partial_1 \partial_2$ by saying that the pseudodifferential or singular integral operator

$$\frac{\partial_1 \partial_2}{\Delta} \tag{2.20}$$

maps L^∞ to BMO, and this holds for similar operators (of order 0) much more generally (as in [GarcR, Garn, Jou, Ste2]). This will be discussed a bit further in Appendix A. Note that the nonlinear problem in Section 1 has a natural linearization which falls into this rubric. (See Appendix A.)

Sobolev embeddings provide another class of linear problems in which BMO comes up naturally. One might wish that a function g on $\mathbf{R}^n$ that satisfies $\nabla g \in L^n(\mathbf{R}^n)$ (in the sense of distributions) were bounded or continuous, but neither of these are true in general, when $n > 1$. However, such a function g is always in BMO, and in the subspace VMO ("vanishing mean oscillation"), in which the measurements of mean oscillation (as in the left side of (2.15) when $n = 2$) tend to 0 as the radius r goes to 0. This is a well-known analogue of continuity in the context of BMO. (See [BrezN, GarcR, Garn, Sem12, Ste2].)

BMO arises in a lot of nonlinear problems, in addition to the one in Section 1. For instance, there are circumstances in which one might wish that the derivative of a conformal mapping in the complex plane were bounded, and it is not, but there are natural estimates in terms of BMO. More precisely, it is BMO for the *logarithm* of the derivative that comes up most naturally. This is closely related to BMO conditions for tangents to curves under certain geometric conditions. See [CoiMe1, CoiMe2, CoiMe3, Davi1, JerK1, Pom1, Pom2, Pom3], for instance. Some basic computations related to the latter were given in Section 1, near the end. In general dimensions (larger than 1), BMO shows up naturally as the logarithm of the density for harmonic measure for Lipschitz domains, and for the logarithm of Jacobians of quasiconformal mappings. See [Dah1, Dah2, JerK2, Geh3, Rei, Ste2] and the references therein. In all dimensions, there are interesting classes of "weights", positive functions which one can use as densities for modifications of Lebesgue measure, whose logarithms lie in BMO, and which in fact correspond to open subsets of BMO (for real-valued functions). These weights have good properties concerning L^p boundedness of singular integral and other operators, and they also show up in other situations, in connection with conformal mappings in the plane, harmonic measure, and Jacobians of quasiconformal mappings

in particular, as above. See [GarcR, Garn, Jou, Ste2, StrT] for information about these classes of weights.

There is a simple reason for BMO functions to arise frequently as some kind of logarithm. In many nonlinear problems there is a symmetry which permits one to multiply some quantity by a constant without changing anything in a significant way. (E.g., think of rescaling or rotating a domain, or a mapping, or multiplying a weight by a positive constant.) At the level of the logarithm this invariance is converted into a freedom to add constants, and this is something that BMO accommodates automatically.

To summarize a bit, there are a lot of situations in which one has some function that one would like to be bounded, but it is not, and for which BMO provides a good substitute. One may not expect at first to have to take measure theory into account, but then it comes up on its own, or works in a natural or reasonable way.

Before leaving this section, let us return to the John–Nirenberg theorem, i.e., the exponential decay estimate in (2.19). How might one try to prove this? The first main point is that one cannot prove (2.19) for a particular disk D using only a bound like (2.15) for that one disk. That would only give a rate of decay on the order of $1/\lambda$. Instead one uses (2.15) over and over again, for many different disks.

Here is a basic strategy. Assume that g is a BMO function with $\|g\|_* \le 1$. First use (2.15) for D itself (with $k = 1$) to obtain that the set of points x in D such that

$$|g(x) - \text{Average}_D(g)| \ge 10, \tag{2.21}$$

is pretty small (in terms of probability). On the bad set where this happens, try to make a good covering by smaller disks on which one can apply the same type of argument. The idea is to then show that the set of points x in D which satisfy

$$|g(x) - \text{Average}_D(g)| \ge 10 + 10 \tag{2.22}$$

is significantly smaller still, and by a definite proportion. If one can repeat this forever, then one can get exponential decay as in (2.19). More precisely, at each stage the size of the deviation of $g(x)$ from $\text{Average}_D(g)$ will increase by the *addition* of 10, while the decrease in the measure of the bad set will decrease *multiplicatively*.

This strategy is roughly correct in spirit, but to carry it out one has to be more careful in the choice of "bad" set at each stage, and in the transition

from one stage to the next. In particular, one should try to control the difference between the average of g over one disk and over one of the smaller disks created in the next step of the process. As a practical matter, it is simpler to work with cubes instead of disks, for the way that they can be decomposed evenly into smaller pieces. The actual construction used is the "Calderón–Zygmund decomposition", which itself has a lot of other applications. See [JohN, GarcR, Garn, Jou, Ste2, Sem12] for more information.

3 Finite polyhedra and combinatorial parameterization problems

Let us now forget about measure theory for the time being, and look at a problem which is, in principle, purely combinatorial.

Fix a positive integer d, and let P be a d-dimensional polyhedron. We assume that P is a finite union of d-dimensional simplices, so that P has "pure" dimension d (i.e., with no lower-dimensional pieces sticking off on their own).

Problem 3.1 *How can one tell if P is a PL (piecewise-linear) manifold? In other words, when is P locally PL-equivalent to $\mathbf{R}^d$ at each point?*

To be precise, P is locally PL-equivalent to $\mathbf{R}^d$ at a point $x \in P$ if there is a neighborhood of x in P which is homeomorphic to an open set in $\mathbf{R}^d$ through a mapping which is piecewise-linear.

This is really just a particular example of a general issue, concerning existence and complexity of parameterizations of a given set. Problem 3.1 has the nice feature that finite polyhedra and piecewise-linear mappings between them can, in principle, be described in finite terms.

Before we try to address Problem 3.1 directly, let us review some preliminary matters. It will be convenient to think of P as being like a simplicial complex, so that it is made up of simplices which are always either disjoint or meet in a whole face of some (lower) dimension. Thus we can speak about the vertices of P, the edges, the 2-dimensional faces, and so on, up to the d-dimensional faces.

Since P is a finite polyhedron, its local structure at any point is pretty simple. Namely, P looks like a cone over a $(d-1)$-dimensional polyhedron at every point. To make this precise, imagine that Q is some finite polyhedron

in some $\mathbf{R}^n$, and let z be a point in $\mathbf{R}^n$ which is affinely-independent of Q, i.e., which lies in the complement of an (affine) plane that contains Q. (We can always replace $\mathbf{R}^n$ with $\mathbf{R}^{n+1}$, if necessary, to ensure that there is such a point.) Let $c(Q)$ denote the set which consists of all rays in $\mathbf{R}^n$ which emanate from z and pass through an element of Q. We include z itself in each of these rays. This defines the "cone over Q centered at z". It does not really depend on the choice of z, in the sense that a different choice of z leads to a set which is equivalent to the one just defined through an invertible affine transformation.

If x is a "vertex" of P, in the sense described above, then there is a natural way to choose a $(d-1)$-dimensional polyhedron Q so that P is the same as the cone over Q centered at x in a neighborhood of x. Let us call Q the *link* of P at x. (Actually, with this description Q is only determined up to piecewise-linear equivalence, but this is adequate for our purposes.)

Now suppose that x is not a vertex. One can still realize P as a cone over a $(d-1)$-dimensional polyhedron near x, but one can also do something more precise. If x is not a vertex, then there is a positive integer k and a k-dimensional face F of P such that x lies in the interior of F. In this case there is a $(d-k-1)$-dimensional polyhedron Q such that P is locally equivalent to $\mathbf{R}^k \times c(Q)$ near x, with x in P corresponding to a point (y, z) in $\mathbf{R}^k \times c(Q)$, where z is the center of $c(Q)$. This same polyhedron Q works for all the points in the interior of F, and we call Q the *link* of F.

Basic Fact 3.2 *P is everywhere locally equivalent to $\mathbf{R}^d$ if and only if all of the various links of P (of all dimensions) are piecewise-linearly equivalent to standard spheres (of the same dimension).*

Here the "standard sphere of dimension m" can be taken to be the boundary of the standard $(m+1)$-dimensional simplex.

Basic Fact 3.2 is standard and not hard to see. The "if" part is immediate, since one knows exactly what the cone over a standard sphere looks like, but for the converse there is a bit more to check. A useful observation is that if Q is a j-dimensional polyhedron whose cone $c(Q)$ is piecewise-linearly equivalent to $\mathbf{R}^{j+1}$ in a neighborhood of the center of $c(Q)$, then Q must be piecewise-linearly equivalent to a standard j-dimensional sphere. This is pretty easy to verify, and one can use it repeatedly for the links of P of codimension larger than 1. (A well-known point here is that one should be careful not to use *radial* projections to investigate links around vertices, but

suitable *pseudo-radial* projections, to fit with the piecewise-linear structure, and not just the topological structure.)

A nice feature of Basic Fact 3.2 is that it sets up a natural induction in the dimensions, since the links of P always have dimension less than P. This leads to the following question.

Problem 3.3 *If Q is a finite polyhedron which is a k-dimensional PL manifold, how can one tell if Q is a PL sphere of dimension k?*

It is reasonable to assume here that Q is a PL-manifold, because of the way that one can use Basic Fact 3.2 and induction arguments.

Problem 3.3 is part of the matter of the Poincaré conjecture, which would seek to say that Q is a PL sphere as soon as it is homotopy-equivalent to a sphere. This has been established in all dimensions except 3 and 4. (Compare with [RouS].) In dimension 4 the Poincaré conjecture was settled by M. Freedman [Fre] in the "topological" category (with ordinary homeomorphisms (continuous mappings with continuous inverses) and topological manifolds), but it remains unknown in the PL case. The PL case is equivalent to the smooth version in this dimension, and both are equivalent to the ordinary topological version in dimension 3. (A brief survey related to these statements is given in Section 8.3 of [FreQ].) Although the Poincaré conjecture is known to hold in the PL category in all higher dimensions (than 4), it does not always work in the smooth category, because of exotic spheres (as in [Mil1, KerM]).

If the PL version of the Poincaré conjecture is true in all dimensions, then this would give one answer to the question of recognizing PL manifolds among finite polyhedra in Problem 3.1. Specifically, our polyhedron P would be a PL manifold if and only if its links are all homotopy-equivalent to spheres (of the correct dimension).

This might seem like a pretty good answer, but there are strong difficulties concerning complexity for matters of homotopy. In order for a k-dimensional polyhedron Q to be a homotopy sphere, it has to be simply connected in particular, at least when $j \geq 2$. In other words, it should be possible to continuously deform any loop in Q to a single point, or, equivalently, to take any continuous mapping from a circle into Q and extend it to a continuous mapping from a closed disk into Q. This extension can entail enormous complexity, in the sense that the filling to the disk might have to be of much greater complexity than the original loop itself.

This is an issue whose geometric significance is often emphasized by Gromov. To describe it more precisely it is helpful to begin with some related algebraic problems, concerning finitely-presented groups.

Let G be a group. A finite presentation of G is given by a finite list $g_1, g_2, \ldots, g_n$ of generators for G together with a finite set $r_1, r_2, \ldots, r_m$ of "relations". The latter are (finite) words made out of the g_i's and their inverses. Let us assume for convenience that the set of relations includes the inverses of all of its elements, and also the empty word. The r_j's are required to be trivial, in the sense that they represent the identity element of G. This implies that arbitrary products of conjugates of the r_j's also represent the identity element, and the final requirement is that if w is any word in the g_i's and their inverses which represents the identity element in G, then it should be possible to obtain w from some product of conjugates of the r_j's through cancellations of subwords of the form $g_i^{-1} g_i$ and $g_i g_i^{-1}$.

For instance, the group $\mathbf{Z}^2$ can be described by two generators a, b and one relation, $aba^{-1}b^{-1}$. As another concrete example, there is the (Baumslag–Solitar) group with two generators x, y and one relation $x^2 y x^{-1} y^{-1}$.

Suppose that a group G and finite presentation of G are given and fixed, and let w be a word in the generators of G and their inverses. Given this information, how can one decide whether w represents the identity element in G? This is called "the word problem" (for G). It is a famous result that there exist finite presentations of groups for which there is no algorithm to solve the word problem. (See [Man].)

To understand what this really means, let us first notice that the set of trivial words for the given presentation is "recursively enumerable". This means that there is an algorithm for listing all of the trivial words. To do this, one simply has to have the algorithm systematically generate all possible conjugates of the relations, all possible products of conjugates of relations, and all possible words derived from these through cancellations as above. In this way the algorithm will constantly generate trivial words, and every trivial word will eventually show up on the list.

However, this does not give a finite procedure for determining that a given word is *not* trivial. A priori one cannot conclude that a given word is not trivial until one goes through the entire list of trivial words.

The real trouble comes from the cancellations. In order to establish the triviality of a given word w, one might have to make derivations through words which are enormously larger, with a lot of collapsing at the end. If one had a bound for the size of the words needed for at least one derivation of the

triviality of a given word w, a bound in terms of an effectively computable (or "recursive") function of the length of w, then the word problem would be algorithmically solvable. One could simply search through all derivations of at most a computable size.

This would not be very efficient, but it would be an algorithm. As it is, even this does not always work, and there are finitely-presented groups for which the derivations of triviality may need to involve words of nonrecursive size compared to the given word.

One should keep in mind that for a given group and a given presentation there is always *some* function $f(n)$ on the positive integers so that trivial words of length at most n admit derivations of their triviality through words of size no greater than $f(n)$. This is true simply because there are only finitely many words of size at most n, and so one can take $f(n)$ to be the maximum size incurred in some finite collection of derivations. The point is that such a function f may not be bounded by a recursive function. This means that f could be really huge, larger than any tower of exponentials, for instance.

The same kind of phenomenon occurs geometrically, for deciding whether a loop in a given polyhedron can be continuously contracted to a point. This is because any finite presentation of a group G can be coded into a finite polyhedron, in such a way that the group G is represented by the fundamental group of the polyhedron. This is a well-known construction in topology.

Note that while the fundamental group of a space is normally defined in terms of *continuous* (based) loops in the space and the continuous deformations between them, in the case of finite polyhedra it is enough to consider *polygonal* loops and deformations which are piecewise-linear (in addition to being continuous). This is another standard fact, and it provides a convenient way to think about complexity for loops and their deformations.

Although arbitrary finite presentations can be coded into finite polyhedra, as mentioned above, this is not the same as saying that they can be coded into compact *manifolds*. It turns out that this does work when the dimension is at least 4, i.e., for each $n \geq 4$ it is true that every finite presentation can be coded into a compact PL manifold of dimension n. This type of coding can be used to convert algorithmic unsolvability results for problems in group theory into algorithmic unsolvability statements in topology. For instance, there does not exist an algorithm to decide when a given finite presentation for a group actually defines the trivial group, and, similarly, there does not exist

an algorithm for deciding when a given manifold (of dimension at least 4) is simply-connected. See [BooHP, Mark1, Mark2, Mark3] for more information and results.

Let us mention that in dimensions 3 and less, it is *not* true that arbitrary finitely-presented groups can be realized as fundamental groups of compact manifolds. Fundamental groups of manifolds are very special in dimensions 1 and 2, as is well known. The situation in dimension 3 is more complicated, but there are substantial restrictions on the groups that can arise as fundamental groups. As an aspect of this, one can look at restrictions related to Poincaré duality. In a different vein, the fundamental group of a 3-dimensional manifold has the property that all of its finitely-generated subgroups are finitely-presented. See [Sco], and Theorem 8.2 on p70 of [Hem1]. See also [Jac]. In another direction, there are relatively few abelian groups which can arise as subgroups of fundamental groups of 3-dimensional manifolds. See [Eps, EvaM], Theorems 9.13 and 9.14 on p84f of [Hem1], and p67-9 of [Jac]. At any rate, it is a large open problem to know exactly what groups arise as fundamental groups of 3-dimensional manifolds.

See also [Thu] and Chapter 12 of [Eps+] concerning these groups. The book [Eps+] treats a number of topics related to computability and groups, and not just in connection with fundamental groups of 3-manifolds. This includes broad classes of groups for which positive results and methods are available. See [Far] as well in this regard.

Beginning in dimension 5, it is known that there is no algorithm for deciding when a compact PL manifold is piecewise-linearly equivalent to a standard (PL) sphere. This is a result of S. Novikov. See Section 10 of [VolKF], and also the appendix to [Nab]. (Note that in dimensions less than or equal to 3, such algorithms do exist. This is classical for dimensions 1, 2; see [Rub1, Rub2, Tho] concerning dimension 3, and related problems and results.) Imagine that we have a PL manifold M of some dimension n whose equivalence to a standard sphere is true but "hard" to check. According to the solution of the Poincaré conjecture in these dimensions, M will be equivalent to an n-sphere if it is homotopy-equivalent to $\mathbf{S}^n$. For standard reasons of algebraic topology, this will happen exactly when M is simply-connected and has trivial homology in dimensions 2 through $n-1$. (Specifically, this uses Theorem 9 and Corollary 24 on pages 399 and 405, respectively, of [Spa]. It also uses the existence of a degree-1 mapping from M to $\mathbf{S}^n$ to get started (i.e., to have a mapping to which the aforementioned results can be applied), and the fact that the homology of M and $\mathbf{S}^n$ vanish in dimensions larger

than n, and are equal to $\mathbf{Z}$ in dimension n. To obtain the degree-1 mapping from M to $\mathbf{S}^n$, one can start with any point in M and a neighborhood of that point which is homeomorphic to a ball. One then collapses the complement of that neighborhood to a point, which gives rise to the desired mapping.) The vanishing of homology can be determined algorithmically, and so if the equivalence of M with an n-sphere is "hard" for algorithmic verification, then the problem must occur already with the simple-connectivity of M. (Concerning this statement about homology, see Appendix E.)

To determine whether M is simply-connected it is enough to check that a finite number of loops in M can be contracted to a point, i.e., some collection of generators for the fundamental group. If this is "hard", then it means that the complexity of the contractions should be enormous compared to the complexity of M. For if there were a bound in terms of a recursive function, then one could reverse the process and use this to get an algorithm which could decide whether M is PL equivalent to a sphere, and this is not possible.

If M is a hard example of a PL manifold which is equivalent to an n-sphere, then any mapping from M to the sphere which realizes this equivalence must necessarily be of very high complexity as well. Because of the preceding discussion, this is also true for mappings which are homotopy-equivalences, or even which merely induce isomorphisms on π_1, if one includes as part of the package of data enough information to justify the condition that the induced mapping on π_1 be an isomorphism. (For a homotopy equivalence, for instance, one could include the mapping f from M to the n-sphere, a mapping g from the n-sphere to M which is a homotopy-inverse to f, and mappings which give homotopies between $f \circ g$ and $g \circ f$ to the identity on the n-sphere and M, respectively.) This is because one could use the mapping to reduce the problem of contracting a loop in M to a point to the corresponding problem for the n-sphere, where the matter of bounds is straightforward.

Similar considerations apply to the problem of deciding when a finite polyhedron P is a PL manifold. Indeed, given a PL manifold M whose equivalence to a sphere is in question, one can use it to make a new polyhedron P by taking the "suspension" of M. This is defined by taking two points y and z which lie outside of a plane that contains M, and then taking the union of all of the (closed) line segments that go from either of y or z to a point in M. One should also be careful to choose y and z so that these line segments never meet, except in the trivial case of line segments from y and z to the same point x in M, with x being the only point of intersection

of the two segments. (One can imagine y and z as lying on "opposite sides" of an affine plane that contains M.)

If M is equivalent to a sphere, then this operation of suspension produces a PL manifold equivalent to the sphere of 1 larger dimension, as one can easily check. If M is not PL equivalent to a sphere, then the suspension P of M is not a PL manifold at all. This is because M is the link of P at the vertices y and z, by construction, so that one is back to the situation of Basic Fact 3.2.

Just as there are PL manifolds M whose equivalence with a sphere is hard, the use of the suspension shows that there are polyhedra P for which the property of being a PL manifold is hard to establish. Through the type of arguments outlined above, when PL coordinates exist for a polyhedron P, they may have to be of enormous complexity compared to the complexity of P itself. This works more robustly than just for PL coordinates, i.e., for any information which is strong enough to give the simple-connectivity of the links of P. Again, this follows the discussion above.

We have focussed on piecewise-linear coordinates for finite polyhedra for the sake of simplicity, but similar themes of complexity come up much more generally, and in a number of different ways. In particular, existence and complexity of parameterizations is often related in a strong manner to the behavior of something like π_1, sometimes in a localized form, as with the links of a polyhedron. For topology of manifolds in high dimensions, π_1 and the filling of loops with disks comes up in the Whitney lemma, for instance. This concerns the separation of crossings of submanifolds through the use of embedded 2-dimensional disks, and it can be very useful for making some geometric constructions. (A very nice brief review of some of these matters is given in Section 1.2 of [DonK].) Localized π_1-type conditions play a crucial role in taming theorems in geometric topology. Some references related to this are [Bin5, Bin6, Bin8, Bur, BurC, Can1, Can2, Dave1, Dave2, Edw1, Moi, Rus1, Rus2].

In Appendix C, we shall review some aspects of geometric topology and the existence and behavior of parameterizations, and the role of localized versions of fundamental groups in particular.

As another type of example, one has the famous "double suspension" results of Edwards and Cannon [Can1, Can3, Dave2, Edw2]. Here one starts with a finite polyhedron H which is a manifold with the same homology as a sphere of the same dimension, and one takes the suspension (described above) of the suspension of H to get a new polyhedron K. The result is

that K is actually homeomorphic to a sphere. A key point is that H is *not* required to be simply-connected. When $\pi_1(H) \neq 0$, it is not possible for the homeomorphism from K to a standard sphere to be piecewise-linear, or even Lipschitz (as in (1.7)). Concerning the latter, see [SieS]. Not much is known about the complexity of the homeomorphisms in this case. (We shall say a bit more about this in Section 5 and Subsection C.5.)

Note that if J is obtained as a single suspension of H, and if $\pi_1(H) \neq 0$, then J cannot be a topological manifold at all (at least if the dimension of H is at least 2). Indeed, if M is a topological manifold of dimension n, then for every point p in M there are arbitrarily small neighborhoods U of p which are homeomorphic to an open n-ball, and $U \backslash \{p\}$ must then be simply-connected when $n \geq 3$. This cannot work for the suspension J of H when $\pi_1(H) \neq 0$, with p taken to be one of the two cone points introduced in the suspension construction.

However, J has the advantage over H that it *is* simply-connected. This comes from the process of passing to the suspension (and the fact that H should be connected, since it has the same homology as a sphere). It is for this reason that the cone points of K do not have the same trouble as in J itself, with no small deleted neighborhoods which are simply-connected. The singularities at the cone points in J lead to trouble with the codimension-2 links in K, but this turns out not to be enough to prevent K from being a topological manifold, or a topological sphere. It *does* imply that the homeomorphisms involved have to distort distances in a very strong way, as in [SieS].

In other words, local homeomorphic coordinates for K do exist, but they are necessarily much more complicated than PL homeomorphisms, even though K is itself a finite polyhedron. As above, there is also a global homeomorphism from K to a sphere. The first examples of finite polyhedra which are homeomorphic to each other but not piecewise-linearly equivalent were given by Milnor [Mil2]. See also [Sta2]. This is the "failure of the Hauptvermutung" (in general). These polyhedra are not PL manifolds, and it turns out that there are examples of compact PL manifolds which are homeomorphic but not piecewise-linearly equivalent too. See [Sie2] for dimensions 5 and higher, and [DonK, FreQ] for dimension 4. In dimensions 3 and lower, this does not happen [Moi, Bin6]. The examples in [Mil2, Sta2, Sie2] involved non-PL homeomorphisms whose behavior is much milder than in the case of double-suspension spheres. There are general results in this direction for PL manifolds (and more broadly) in dimensions greater than or equal to 5. See

[Sul1, SieS]. Analogous statements fail in dimension 4, by [DonS].

Some other examples where homeomorphic coordinates do not exist, or necessarily have complicated behavior, even though the geometry behaves well in other ways, are given in [Sem7, Sem8].

See [DaviS4, HeiS, HeiY, MülŠ, Sem3, Tor1, Tor2] for some related topics concerning homeomorphisms and bounds for their behavior.

One can try to avoid difficulties connected to π_1 by using mappings with branching rather than homeomorphisms. This is discussed further in Appendix B.

Questions of algorithmic undecidability in topology have been revisited in recent years, in particular by Nabutovsky and Weinberger. See [NabW1, NabW2], for instance, and the references therein.

4 Quantitative topology, and calculus on singular spaces

One of the nice features of Euclidean spaces is that it is easy to work with functions, derivatives, and integrals. Here is a basic example of this. Let f be a real-valued function on $\mathbf{R}^n$ which is continuously differentiable and has compact support, and fix a point $x \in \mathbf{R}^n$. Then

$$|f(x)| \le \frac{1}{\nu_n} \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-1}} \, |\nabla f(y)| \, dy, \tag{4.1}$$

where ν_n denotes the $(n-1)$-dimensional volume of the unit sphere in $\mathbf{R}^n$, and dy refers to ordinary n-dimensional volume.

This inequality provides a way to say that the values of a function are controlled by *averages* of its derivative. In this respect it is like Sobolev and isoperimetric inequalities, to which we shall return in a moment.

To prove (4.1) one can proceed as follows (as on p125 of [Ste1]). Let v be any element of $\mathbf{R}^n$ with $|v| = 1$. Then

$$f(x) = -\int_0^\infty \frac{\partial}{\partial t} f(x + tv) \, dt, \tag{4.2}$$

by the fundamental theorem of calculus. Thus

$$|f(x)| \le \int_0^\infty |\nabla f(x + tv)| \, dt. \tag{4.3}$$

This is true for every v in the unit sphere of $\mathbf{R}^n$, and by averaging over these v's one can derive (4.1) from (4.3).

To put this into perspective, it is helpful to look at a situation where analogous inequalities make sense but fail to hold. Imagine that one is interested in inequalities like (4.1), but for 2-dimensional surfaces in $\mathbf{R}^3$ instead of Euclidean spaces themselves. Let S be a smoothly embedded 2-dimensional submanifold of $\mathbf{R}^3$ which looks like a 2-plane with a bubble attached to it. Specifically, let us start with the union of a 2-plane P and a standard (round) 2-dimensional sphere Σ which is tangent to P at a single point z. Then cut out a little neighborhood of z, and glue in a small "neck" as a bridge between the plane and the sphere to get a smooth surface S.

If the neck in S is very small compared to the size of Σ, then this is bad for an inequality like (4.1). Indeed, let x be the point on Σ which is exactly opposite from z, and consider a smooth function f which is equal to 1 on most of Σ (and at x in particular) and equal to 0 on most of P. More precisely, let us choose f so that its gradient is concentrated near the bridge between Σ and P. If f makes the transition from vanishing to being 1 in a reasonable manner, then the integral of $|\nabla f|$ on S will be very small. This is not hard to check, and it is bad for having an inequality like (4.1), since the left-hand side would be 1 and the right-hand side would be small. In particular, one could not have uniform bounds that would work for arbitrarily small bridges between P and Σ.

The inequality (4.1) is a relative of the usual Sobolev and isoperimetric inequalities, which say the following. Fix a dimension n again, and an exponent p that satisfies $1 \le p < n$. Define q by $1/q = 1/p - 1/n$, so that $p < q < \infty$. The Sobolev inequalities assert the existence of a constant $C(n, p)$ such that

$$\Big(\int_{\mathbf{R}^n} |f(x)|^q \, dx \Big)^{\frac{1}{q}} \le C(n, p) \Big(\int_{\mathbf{R}^n} |\nabla f(x)|^p \, dx \Big)^{\frac{1}{p}} \tag{4.4}$$

for all functions f on $\mathbf{R}^n$ that are continuously differentiable and have compact support. One can also allow more general functions, with ∇f interpreted in the sense of distributions.

The *isoperimetric inequality* states that if D is a domain in $\mathbf{R}^n$ (which is bounded and has reasonably smooth boundary, say), then

$$\begin{aligned} &n\text{-dimensional volume of } D \\ &\qquad \le C(n) \, ((n-1)\text{-dimensional volume of } \partial D)^{\frac{n}{n-1}}. \end{aligned} \tag{4.5}$$

This is really just a special case of (4.4), with $p = 1$ and f taken to be the characteristic function of D (i.e., the function that is equal to 1 on D and 0 on the complement of D). In this case ∇f is a (vector-valued) measure, and the right-hand side of (4.4) should be interpreted accordingly. Conversely, Sobolev inequalities for all p can be derived from isoperimetric inequalities, by applying the latter to sets of the form

$$\{x \in \mathbf{R}^n : |f(x)| > t\}, \tag{4.6}$$

and then making suitable integrations in t.

The sharp version of the isoperimetric inequality states that (4.5) holds with the constant $C(n)$ that gives equality in the case of a ball. See [Fed]. One can also determine sharp constants for (4.4), as on p39 of [Aub].

Note that the choice of the exponent $n/(n-1)$ in the right side of (4.5) is determined by scaling considerations, i.e., in looking what happens to the two sides of (4.5) when one dilates the domain D by a positive factor. The same is true of the relationship between p and q in (4.4), and the power $n-1$ in the kernel on the right side of (4.1).

The inequality (4.1) is a basic ingredient in one of the standard methods for proving Sobolev and isoperimetric inequalities (but not necessarily with sharp constants). Roughly speaking, once one has (4.1), the rest of the argument works at a very general level of integral operators on measure spaces, rather than manifolds and derivatives. This is not quite true for the $p = 1$ case of (4.4), for which the general measure-theoretic argument gives a slightly weaker result. See Chapter V of [Ste1] for details. The slightly weaker result does give an isoperimetric inequality (4.5), and it is not hard to recover the $p = 1$ case of (4.4) from the weaker version using a bit more of the localization properties of the gradient than are kept in (4.1). (See also Appendix C of [Sem9], especially Proposition C.14.)

The idea of these inequalities makes sense much more broadly than just on Euclidean spaces, but they may not always work very well, as in the earlier example with bubbling. To consider this further, let M be a smooth Riemannian manifold of dimension n, and let us assume for simplicity that M is unbounded (like $\mathbf{R}^n$). Let us also think of M as coming equipped with a distance function $d(x, y)$ with the usual properties ($d(x, y)$ is nonnegative, symmetric in x and y, vanishes exactly when $x = y$, and satisfies the triangle inequality). One might take $d(x, y)$ to be the geodesic distance associated to the Riemannian metric on M, but let us not restrict ourselves to this

case. For instance, imagine that M is a smooth submanifold of some higher-dimensional $\mathbf{R}^k$, and that $d(x, y)$ is simply the ambient Euclidean distance $|x - y|$ inherited from $\mathbf{R}^k$. In general this could be much smaller than the geodesic distance.

We shall make the standing assumption that the distance $d(x, y)$ and the Riemannian geodesic distance are approximately the same, each bounded by twice the other, on sufficiently small neighborhoods about any given point in M. This ensures that $d(x, y)$ is compatible with quantities defined locally on M using the Riemannian metric, like the volume measure, and the length of the gradient of a function. Note that this local compatibility condition for the distance function $d(x, y)$ and the Riemannian metric is satisfied automatically in the situation mentioned above, where M is a submanifold of a larger Euclidean space and $d(x, y)$ is inherited from the ambient Euclidean distance. We shall also require that the distance $d(x, y)$ be compatible with the (manifold) topology on M, and that it be complete. This prevents things like infinite ends in M which asymptotically approach finite points in M with respect to $d(x, y)$.

The smoothness of M should be taken in the character of an a priori assumption, with the real point being to have bounds that do not depend on the presence of the smoothness in a quantitative way. Indeed, the smoothness of M will not really play an essential role here, but will be convenient, so that concepts like volume, gradient, and lengths of gradients are automatically meaningful.

Suppose for the moment that M is *bilipschitz equivalent* to $\mathbf{R}^n$ equipped with the usual Euclidean metric. This means that there is a mapping ϕ from $\mathbf{R}^n$ onto M and a constant k such that

$$k^{-1}|z - w| \le d(\phi(z), \phi(w)) \le k\,|z - w| \qquad \text{for all } z, w \in \mathbf{R}^n. \tag{4.7}$$

In other words, ϕ should neither expand or shrink distances by more than a bounded amount. This implies that ϕ does not distort the corresponding Riemannian metrics or volume by more than bounded factors either, as one can readily show. In this case the analogues of (4.1), (4.4), and (4.5) all hold for M, with constants that depend only on the constants for $\mathbf{R}^n$ and the distortion factor k. This is because any test of these inequalities on M can be "pulled back" to $\mathbf{R}^n$ using ϕ, with the loss of information in moving between M and $\mathbf{R}^n$ limited by the bilipschitz condition for ϕ.

This observation helps to make clear the fact that inequalities like (4.1), (4.4), and (4.5) do not really require much in the way of smoothness for

the underlying space. Bounds on curvature are not preserved by bilipschitz mappings, just as bounds on higher derivative of functions are not preserved. Bilipschitz mappings can allow plenty of spiralling and corners in M (or approximate corners, since we are asking that M that be smooth a priori).

Although bilipschitz mappings are appropriate here for the small amount of regularity involved, the idea of a "parameterization" is too strong for the purposes of inequalities like (4.1), (4.4), and (4.5). One might say that these inequalities are like algebraic topology, but more quantitative, while parameterizations are more like homeomorphisms, which are always more difficult. (Some other themes along these lines will be discussed in Appendix D. Appendix C is related to this as well. See also [HanH].)

I would like to describe now some conditions on M which are strong enough to give bounds as in (4.1), but which are quite a bit weaker than the existence of a bilipschitz parameterization. First, let us explicitly write down the analogue of (4.1) for M. If x is any element of M, this analogue would say that there is a constant C so that

$$|f(x)| \le C \int_M \frac{1}{d(x,y)^{n-1}} \, |\nabla f(y)| \, dVol(y) \tag{4.8}$$

for all continuously differentiable functions f on M, where $|\nabla f(y)|$ and the volume measure $dVol(y)$ are defined in terms of the Riemannian structure that comes with M.

The next two definitions give the conditions on M that we shall consider. These and similar notions have come up many times in various parts of geometry and analysis, as in [Ale, AleV2, AleV3, As1, As2, As3, CoiW1, CoiW2, Gro1, Gro2, HeiKo1, HeiKo2, HeiKo2, HeiY, Pet1, Pet2, Väi6].

Definition 4.9 (The doubling condition) *A metric space $(M, d(x,y))$ is said to be* doubling *(with constant L_0) if each ball B in M with respect to $d(x,y)$ can be covered by at most L_0 balls of half the radius of B.*

Notice that Euclidean spaces are automatically doubling, with a constant L_0 that depends only on the dimension. Similarly, every subset of a (finite-dimensional) Euclidean space is doubling, with a uniform bound for its doubling constant.

Definition 4.10 (Local linear contractability) *A metric space $(M, d(x,y))$ is said to be* locally linearly contractable *(with constant L_1) if the following is*

true. Let B be a ball in M with respect to $d(x, y)$, and with radius no greater than L_1^{-1} times the diameter of M. (Arbitrary radii are permitted when M is unbounded, as in the context of the present general discussion.) Then (local linear contractability means that) it should be possible to continuously contract B to a point inside of $L_1 B$, i.e., inside the ball with the same center as B and L_1 times the radius.

This is a kind of quantitative and scale-invariant condition of local contractability. It prevents certain types of cusps or bubbling, for instance. Both this and the doubling condition hold automatically when M admits a bilipschitz parameterization by $\mathbf{R}^n$, with uniform bounds in terms of the bilipschitz constant k in (4.7) (and the dimension for the doubling condition).

Theorem 4.11 *If M and $d(x, y)$ are as before, and if $(M, d(x, y))$ satisfies the doubling and local linear contractability conditions with constants L_0 and L_1, respectively, then (4.8) holds with a constant C that depends only on L_0, L_1, and the dimension n.*

This was proved in [Sem9]. Before we look at some aspects of the proof, some remarks are in order about what the conclusions really mean.

In general one cannot derive bounds for Sobolev and isoperimetric inequalities for M just using (4.8). One might say that (4.8) is only as good as the behavior of the volume measure on M. If the volume measure on M behaves well, with bounds for the measure of balls like ones on $\mathbf{R}^n$, then one can derive conclusions from (4.8) in much the same way as for Euclidean spaces. See Appendices B and C in [Sem9].

The doubling and local linear contractability conditions do not themselves say anything about the behavior of the volume on M, and indeed they tolerate fractal behavior perfectly well. To see this, consider the metric space which is $\mathbf{R}^n$ as a set, but with the metric $|x - y|^\alpha$, where α is some fixed number in $(0, 1)$. This is a kind of abstract and higher-dimensional version of standard fractal snowflake curves in the plane. However, the doubling and local linear contractability conditions work just as well for $(\mathbf{R}^n, |x - y|^\alpha)$ as for $(\mathbf{R}^n, |x - y|)$, just with slightly different constants.

How might one prove Theorem 4.11? It would be nice to be able to mimic the proof of (4.1), i.e., to find a family of rectifiable curves in M which go from x to infinity and whose arclength measures have approximately the same kind of distribution in M as rays in $\mathbf{R}^n$ emanating from a given

point. Such families exist (with suitable bounds) when M admits a bilipschitz parameterization by $\mathbf{R}^n$, and they also exist in more singular circumstances.

Unfortunately, it is not so clear how to produce families of curves like these without some explicit information about the space M in question. This problem was treated in a special case in [DaviS1], with M a certain kind of (nonsmooth) conformal deformation of $\mathbf{R}^n$. The basic idea was to obtain these curves from level sets of certain mappings with controlled behavior. When $n = 2$, for instance, imagine a standard square Q, with opposing vertices p and q. The boundary of Q can be thought of as a pair of paths α, β from p to q, each with two segments, two sides of Q. If τ is a function on Q which equals 0 on α and 1 on β (and is somewhat singular at p and q), then one can try to extract a family of paths from p to q in Q from the level sets

$$\{x \in \mathbf{R}^2 : \tau(x) = t\}, \qquad 0 < t < 1. \tag{4.12}$$

For the standard geometry on $\mathbf{R}^2$ one can write down a good family of curves and a good function τ explicitly. For a certain class of conformal deformations of $\mathbf{R}^2$ one can make constructions of functions τ with approximately the same behavior as in the case of the standard metric, and from these one can get controlled families of curves.

These constructions of functions τ used the standard Euclidean geometry in the background in an important way. For the more general setting of Theorem 4.11 one needs to proceed somewhat differently, and it is helpful to begin with a different formulation of the kind of auxiliary functions to be used.

Given a point x in $\mathbf{R}^n$, there is an associated spherical projection $\pi_x : \mathbf{R}^n \backslash \{x\} \to \mathbf{S}^{n-1}$ given by

$$\pi_x(u) = \frac{u - x}{|u - x|}. \tag{4.13}$$

This projection is topologically nondegenerate, in the sense that it has degree equal to 1. Here the "degree" can be defined by restricting π_x to a sphere around x and taking the degree of this mapping (from an $(n-1)$-dimensional sphere to another one) in the usual sense. (See [Mas, Mil3, Nir] concerning the notion of degree of a mapping.) Also, this mapping satisfies the bound

$$|d\pi_x(u)| \le C\,|u - x|^{-1} \tag{4.14}$$

for all $u \in \mathbf{R}^n \backslash \{x\}$, where $d\pi_x(u)$ denotes the differential of π_x at u, and C is some constant. One can write down the differential of π_x explicitly,

and (4.14) can be replaced by an equality, but this precision is not needed here, and not available in general. The rays in $\mathbf{R}^n$ that emanate from x are exactly the fibers of the mapping π_x, and bounds for the distribution of their arclength measures can be seen as a consequence of (4.14), using the "co-area theorem" [Fed, Morg, Sim].

One can also think of π_x as giving (4.1) in the following manner. Let ω denote the standard volume form on $\mathbf{S}^{n-1}$, a differential form of degree $n-1$, and normalized so that

$$\int_{\mathbf{S}^{n-1}} \omega = 1. \tag{4.15}$$

Let λ denote the differential form on $\mathbf{R}^n \backslash \{0\}$ obtained by pulling ω back using π_x. Then (4.14) yields

$$|\lambda(u)| \le C' \, |u - x|^{-n+1} \tag{4.16}$$

for all $u \in \mathbf{R}^n \backslash \{x\}$, where C' is a slightly different constant from before. In particular, λ is locally integrable across x (and smooth everywhere else). This permits one to take the exterior derivative of λ on all of $\mathbf{R}^n$ in the (distributional) sense of currents [Fed, Morg], and the result is that $d\lambda$ is the current of degree n which is a Dirac mass at x. More precisely, $d\lambda = 0$ away from x because ω is automatically closed (being a form of top degree on $\mathbf{S}^{n-1}$), and because the pull-back of a closed form is always closed. The Dirac mass at x comes from a standard Stokes' theorem computation, which uses the observation that the integral of λ over any $(n-1)$-sphere in $\mathbf{R}^n$ around x is equal to 1. (The latter is one way to formulate the fact that the degree of π_x is 1.)

This characterization of $d\lambda$ as a current on $\mathbf{R}^n$ means that

$$\int_{\mathbf{R}^n} df \wedge \lambda = -f(x) \tag{4.17}$$

when f is a smooth function on $\mathbf{R}^n$ with compact support. This yields (4.1), because of (4.16). (A similar use of differential forms was employed in [DaviS1].)

The general idea of the mapping π_x also makes sense in the context of Theorem 4.11. Let M, $d(y, z)$ be as before, and fix a point x in M. One would like to find a mapping $\pi_x : M \backslash \{x\} \to \mathbf{S}^{n-1}$ which is topologically nondegenerate and satisfies

$$|d\pi_x(u)| \le K \, d(u, x)^{-1} \tag{4.18}$$

for some constant K and all $u \in M \backslash \{x\}$. Note that now the norm of the differential of π_x involves the Riemannian metric on M. For the topological nondegeneracy of π_x, let us ask that it have nonzero degree on small spheres in M that surround x in a standard way. This makes sense, because of the a priori assumption that M be smooth.

If one can produce such a mapping π_x, then one can derive (4.8) as a consequence, using the same kind of argument with differential forms as above. One can also find enough curves in the fibers of π_x, with control on the way that their arclength measures are distributed in M, through the use of co-area estimates. For this the topological nondegeneracy of π_x is needed for showing that the fibers of π_x connect x to infinity in M.

In the context of conformal deformations of $\mathbf{R}^n$, as in [DaviS1], such mappings π_x can be obtained as perturbations to the standard mapping in (4.13). This is described in [Sem10]. For Theorem 4.11, the method of [Sem9] does not use mappings quite like π_x, but a "stabilized" version from which one can draw similar conclusions. In this stabilized version one looks for mappings from M to $\mathbf{S}^n$ (instead of $\mathbf{S}^{n-1}$) which are constant outside of a (given) ball, topologically nontrivial (in the sense of nonzero degree), and which satisfy suitable bounds on their differentials. These mappings are like snapshots of pieces of M, and one has to move them around in a controlled manner. This means moving them both in terms of location (the center of the supporting ball) and scale (the radius of the ball).

At this stage the hypotheses of Theorem 4.11 may make more sense. Existence of mappings like the ones described above is a standard matter in topology, except for the question of uniform bounds. The hypotheses of Theorem 4.11 (the doubling condition and local linear contractability) are also in the nature of quantitative topology. Note, however, that the kind of bounds involved in the hypotheses of the theorem and the construction of mappings into spheres are somewhat different from each other, with bounds on the differentials being crucial for the latter, while control over moduli of continuity does not come up in the former. (The local linear contractability condition restricts the overall distances by which points are displaced in the contractions, but not the sizes of the smaller-scale oscillations, as in a modulus of continuity.) In the end the bounds for the differentials come about because the hypotheses of Theorem 4.11 permit one to reduce various constructions and comparisons to finite models of controlled complexity.

In the proof of Theorem 4.11 there are three related pieces of information that come out, namely (1) estimates for the behavior of functions on our

space M in terms of their derivatives, as in (4.8), (2) families of curves in M which are well-distributed in terms of arclength measure, and (3) mappings to spheres with certain estimates and nondegeneracy properties. These three kinds of information are closely linked, through various dualities, but to some extent they also have their own lives. Each would be immediate if M had a bilipschitz parameterization by $\mathbf{R}^n$, but in fact they are more robust than that, and much easier to verify.

Indeed, one of the original motivations for [DaviS1] was the problem of determining which conformal deformations of $\mathbf{R}^n$ lead to metric spaces (through the geodesic distance) which are bilipschitz equivalent to $\mathbf{R}^n$. The deformations are allowed to be nonsmooth here, but this does not matter too much, because of the natural scale-invariance of the problem, and because one seeks uniform bounds. This problem is the same in essence as asking which (positive) functions on $\mathbf{R}^n$ arise as the Jacobian of a quasiconformal mapping, modulo multiplication by a positive function which is bounded and bounded away from 0.

Some natural necessary conditions are known for these questions, with a principal ingredient coming from [Geh3]. It was natural to wonder whether the necessary conditions were also sufficient. As a test for this, [DaviS1] looked at the Sobolev and related inequalities that would follow if the necessary conditions were sufficient. These inequalities could be stated directly in terms of the data of the problems, the conformal factor or prospective Jacobian. The conclusion of [DaviS1] was that these inequalities could be derived directly from the conditions on the data, independently of whether these conditions were sufficient for the existence of bilipschitz/quasiconformal mappings as above.

In [Sem8] it was shown that the candidate conditions are *not* sufficient for the existence of such mappings, at least in dimensions 3 and higher. (Dimension 2 remains open.) The simplest counterexamples involved considerations of localized fundamental groups, in much the same fashion as in Section 3. (Another class of counterexamples were based on a different mechanism, although these did not start in dimension 3.) These counterexamples are all perfectly well-behaved in terms of the doubling and local linear contractability properties, and in fact are much better than that.

Part of the bottom line here is that spaces can have geometry which behaves quite well for many purposes even if they do not behave so well in terms of parameterizations.

For some other aspects of "quantitative topology", see [Ale, AleV2, AleV3,

Att1, Att2, BloW, ChaF, Che, Fer1, Fer2, Fer3, Fer4, Geh2, Gro1, Gro2, HeiY, HeiS, Luu, Pet1, Pet2, TukV, Väi3, Väi5, Väi6]. Related matters of Sobolev and other inequalities on non-smooth spaces come up in [HeiKo2, HeiKo2, HeiKo+], in connection with the behavior of quasiconformal mappings.

5 Uniform rectifiability

A basic fact in topology is that there are spaces which are *manifold factors* but not manifolds. That is, there are topological spaces M such that $M \times \mathbf{R}$ is a manifold (locally homeomorphic to a Euclidean space) but M is not. This can even happen for finite polyhedra, because of the double-suspension results of Edwards and Cannon. See [Dave2, Edw2, Kir] for more information and specific examples. (We shall say a bit more about this in Appendix C.)

Uniform rectifiability is a notion of controlled geometry that trades topology for estimates. It tolerates some amount of singularities, like holes and crossings, and avoids some common difficulties with homeomorphisms, such as manifold factors.

The precise definition is slightly technical, and relies on measure theory in a crucial way. In many respects it is analogous to the notion of BMO from Section 2. The following is a preliminary concept that helps to set the stage.

Definition 5.1 (Ahlfors regularity) *Fix n and d, with n a positive integer and $0 < d \le n$. A set E contained in $\mathbf{R}^n$ is said to be* (Ahlfors) regular of dimension d *if it is closed, and if there is a positive Borel measure μ supported on E and a constant $C > 0$ such that*

$$C^{-1} r^d \le \mu(B(x, r)) \le C r^d \tag{5.2}$$

for all $x \in E$ and $0 < r \le \operatorname{diam} E$. Here $B(x, r)$ denotes the (open) ball with center x and radius r.

Roughly speaking, this definition asks that E behave like ordinary Euclidean space in terms of the distribution of its mass. Notice that d-planes satisfy this condition automatically, with μ equal to the ordinary d-dimensional volume. The same is true for compact smooth manifolds, and finite polyhedra which are given as unions of d-dimensional simplices (i.e., with no lower-dimensional pieces sticking off in an isolated manner). There are also plenty

of "fractal" examples, like self-similar Cantor sets and snowflake curves. In particular, the dimension d can be any (positive) real number.

A basic fact is that if E is regular and μ is as in Definition 5.1, then μ is practically the same as d-dimensional Hausdorff measure H^d restricted to E. Specifically, μ and H^d are each bounded by constant multiples of the other when applied to subsets of E. This is not hard to prove, and it shows that μ is essentially unique. Definition 5.1 could have been formulated directly in terms of Hausdorff measure, but the version above is a bit more elementary.

Let us recall the definition of a bilipschitz mapping. Let A be a set in $\mathbf{R}^n$, and let f be a mapping from A to some other set in $\mathbf{R}^n$. We say that f is *k-bilipschitz*, where k is a positive number, if

$$k^{-1} \, |x - y| \le |f(x) - f(y)| \le k \, |x - y| \tag{5.3}$$

for all $x, y \in A$.

Definition 5.4 (Uniform rectifiability) *Let E be a subset of $\mathbf{R}^n$ which is Ahlfors regular of dimension d, where d is a positive integer, $d < n$, and let μ be a positive measure on E as in Definition 5.1. Then E is* uniformly rectifiable *if there exists a positive constant k so that for each $x \in E$ and each $r > 0$ with $r \le \operatorname{diam} E$ there is a closed subset A of $E \cap \overline{B}(x, r)$ such that*

$$\mu(A) \ge \frac{9}{10} \cdot \mu(E \cap \overline{B}(x, r)) \tag{5.5}$$

and

$$\text{there is a } k\text{-bilipschitz mapping } f \text{ from } A \text{ into } \mathbf{R}^d. \tag{5.6}$$

In other words, inside of each "snapshot" $E \cap \overline{B}(x, r)$ of E there should be a large subset, with at least 90% of the points, which is bilipschitz equivalent to a subset of $\mathbf{R}^d$, and with a uniform bound on the bilipschitz constant. This is like asking for a controlled parameterization, except that we allow for holes and singularities.

Definition 5.4 should be compared with the classical notion of (countable) rectifiability, in which one asks that E be covered, except for a set of measure 0, by a countable union of sets, each of which is bilipschitz equivalent to a subset of $\mathbf{R}^d$. Uniform rectifiability implies this condition, but it is stronger, because it provides quantitative information at definite scales, while the classical notion really only gives asymptotic information as one zooms in

at almost any point. See [Fal, Fed, Mat] for more information about classical rectifiability.

Normally one would be much happier to simply have bilipschitz coordinates outright, without having to allow for bad sets of small measure where this does not work. In practice bilipschitz coordinates simply do not exist in many situations where one might otherwise hope to have them. This is illustrated by the double-suspension spheres of Edwards and Cannon [Can1, Can3, Dave2, Edw2], and the observations about them in [SieS]. Further examples are given in [Sem7, Sem8].

The use of arbitrary scales and locations is an important part of the story here, and is very similar to the concept of BMO. At the level of a single snapshot, a fixed ball $\overline{B}(x, r)$ centered on E, the bad set may seem pretty wild, as nothing is said about what goes on there in (5.5) or (5.6). However, uniform rectifiability, like BMO, applies to all snapshots equally, and in particular to balls in which the bad set is concentrated. Thus, inside the bad set, there are in fact further controls. We shall see other manifestations of this later, and the same basic principle is used in the John–Nirenberg theorem for BMO functions (discussed in Section 2).

Uniform rectifiability provides a substitute for (complete) bilipschitz coordinates in much the same way that BMO provides a substitute for L^∞ bounds, as in Section 2. Note that L^∞ bounds and bilipschitz coordinates automatically entail uniform control over all scales and locations. This is true just because of the way they are defined, i.e., a bounded function is bounded in all snapshots, and with a uniform majorant. With BMO and uniform rectifiability the scale-invariance is imposed by hand.

It may be a little surprising that one can get anything new through concepts like BMO and uniform rectifiability. For instance, suppose that f is a locally-integrable function on $\mathbf{R}^k$, and that the averages

$$\frac{1}{\omega_k \, t^k} \int_{B(z,t)} |f(w)| \, dw \tag{5.7}$$

are uniformly bounded, independently of z and t. Here ω_k denotes the volume of the unit ball in $\mathbf{R}^k$, so that $\omega_k \, t^k$ is the volume of $B(z, t)$. This implies that f must itself be bounded by the same amount almost everywhere on $\mathbf{R}^k$, since

$$f(u) = \lim_{t \to 0} \frac{1}{\omega_k \, t^k} \int_{B(z,t)} f(w) \, dw \tag{5.8}$$

almost everywhere on $\mathbf{R}^k$. Thus a uniform bound for the size of the snapshots does imply a uniform bound outright. For BMO the situation is different because one asks only for a uniform bound on the mean *oscillation* in every ball. In other words, one also has the freedom to make renormalizations by additive constants when moving from place to place, and this gives enough room for some unbounded functions, like $\log |x|$. Uniform rectifiability is like this as well, although with different kinds of "renormalizations" available.

These remarks might explain why *some* condition like uniform rectifiability could be useful or natural, but why the specific version above in particular? Part of the answer to this is that nearly all definitions of this nature are equivalent to the formulation given above. For instance, the 9/10 in (5.5) can be replaced by any number strictly between 0 and 1. See [DaviS3, DaviS5] for more information.

Another answer lies in a theme often articulated by Coifman, about the way that operator theory can provide a good guide for geometry. One of the original motivations for uniform rectifiability came from the "Calderón program" [Cal2], concerning the L^p-boundedness of certain singular operators on curves and surfaces of minimal smoothness. David [Davi2, Davi3, Davi5] showed that uniform rectifiability of a set E implies L^p-boundedness of wide classes of singular operators on E. (See [Cal1, Cal2, CoiDM, CoiMcM] and the references therein for related work connected to the Calderón program.) In [DaviS3], a converse was established, so that uniform rectifiability of an Ahlfors-regular set E is actually equivalent to the boundedness of a suitable class of singular integral operators (inherited from the ambient Euclidean space $\mathbf{R}^n$). See also [DaviS2, DaviS5, MatMV, MatP].

Here is a concrete statement about uniform rectifiability in situations where well-behaved parameterizations would be natural but may not exist.

Theorem 5.9 *Let E be a subset of $\mathbf{R}^n$ which is regular of dimension d. If E is also a d-dimensional topological manifold and satisfies the local linear contractability condition (Definition 4.10), then E is uniformly rectifiable.*

Note that Ahlfors-regularity automatically implies the doubling condition (Definition 4.9).

Theorem 5.9 has been proved by G. David and myself. Now-a-days we have better technology, which allows for versions of this which are localized to individual "snapshots", rather than using all scales and locations at once. See [DaviS11] (with some of the remarks in Section 12.3 of [DaviS11] helping

to provide a bridge to the present formulation). We shall say a bit more about this, near the end of Subsection 5.3.

The requirement that E be a topological manifold is convenient, but weaker conditions could be used. For that matter, there are natural variations of local linear contractability too.

One can think of Theorem 5.9 and related results in the following terms. Given a compact set K, upper bounds for the d-dimensional Hausdorff measure of K together with lower bounds for the d-dimensional topology of K should lead to strong information about the geometric behavior of K. See [DaviS9, DaviS11, Sem6] for more on this.

To understand better what Theorem 5.9 means, let us begin by observing that the hypotheses of Theorem 5.9 would hold automatically if E were bilipschitz equivalent to $\mathbf{R}^d$, or if E were compact and admitted bilipschitz local coordinates from $\mathbf{R}^d$. Under these conditions, a test of the hypotheses of Theorem 5.9 on E can be converted into a similar test on $\mathbf{R}^d$, where it can then be resolved in a straightforward manner.

A similar argument shows that the hypotheses of Theorem 5.9 are "bilipschitz invariant". More precisely, if F is another subset of $\mathbf{R}^n$ which is bilipschitz equivalent to E, and if the hypotheses of Theorem 5.9 holds for one of E and F, then it automatically holds for the other.

Since the existence of bilipschitz coordinates implies the hypotheses of Theorem 5.9, we cannot ask for more than that in the conclusions. In other words, bilipschitz coordinates are at the high end of what one can hope for in the context of Theorem 5.9. The hypotheses of Theorem 5.9 do in fact rule out a lot of basic obstructions to the existence of bilipschitz coordinates, like cusps, fractal behavior, self-intersections and approximate self-intersections, and bubbles with very small necks. (Compare with Section 4, especially Theorem 4.11 and the discussion of its proof and consequences.) Nonetheless, it can easily happen that a set E satisfies the hypotheses of Theorem 5.9 but does not admit bilipschitz local coordinates. Double-suspension spheres provide spectacular counterexamples for this (using the observations of [SieS]). Additional counterexamples are given in [Sem7, Sem8].

We should perhaps emphasize that the assumption of being a topological manifold in Theorem 5.9 does not involve bounds. By contrast, uniform rectifiability does involve bounds, which is part of the point. In the context of Theorem 5.9, the proof shows that the uniform rectifiability constants for the conclusion are controlled in terms of the constants that are implicit in the hypotheses, i.e., in Ahlfors-regularity, the linear contractability condition,

and the dimension.

If bilipschitz coordinates are at the high end of what one could hope for, what happens if one asks for less? What if one asks for homeomorphic local coordinates with some control, but not as much? For instance, instead of bounding the "rate" of continuity through Lipschitz conditions like

$$|f(x) - f(y)| \le C\,|x - y| \tag{5.10}$$

(for some C and all x, y in the domain of f), one could work with Hölder continuity conditions, which have the form

$$|f(x) - f(y)| \le C'\,|x - y|^{\gamma}. \tag{5.11}$$

Here γ is a positive number, sometimes called the Hölder "exponent". As usual, (5.11) is supposed to hold simultaneously for all x and y in the domain of f, and with a fixed constant C'. When x and y are close to each other and γ is less than 1, this type of condition is strictly weaker than that of being Lipschitz. Just as $f(x) = |x|$ is a standard example of a Lipschitz function that is not differentiable at the origin, $g(x) = |x|^{\gamma}$ is a basic example of a function that is Hölder continuous of order γ, $\gamma \le 1$, but not of any order larger than γ, in any neighborhood of the origin.

Instead of local coordinates which are bilipschitz, one could consider ones that are "bi-Hölder", i.e., Hölder continuous and with Hölder continuous inverse. It turns out that double-suspension spheres do not admit bi-Hölder local coordinates when the Hölder exponent γ lies above an explicit threshold. Specifically, if P is an n-dimensional polyhedron which is the double-suspension of an $(n-2)$-dimensional homology sphere that is not simply connected, then there are points in P (along the "suspension circle") for which bi-Hölder local coordinates of exponent $\gamma > 1/(n-2)$ do not exist. This comes from the same argument as in [SieS]. More precisely, around these points in P, there do not exist homeomorphic local coordinates from subsets of $\mathbf{R}^n$ for which the inverse mapping is Hölder continuous of order $\gamma > 1/(n-2)$ (without requiring a Hölder condition for the mapping itself).

Given any positive number a, there are examples in [Sem8] so that local coordinates (at some points) cannot have their inverses be Hölder continuous of order a. These examples do admit bi-Hölder local coordinates (with a smaller exponent), and even "quasisymmetric" [TukV] coordinates, and they satisfy the hypotheses of Theorem 5.9. In [Sem7] there are examples which satisfy the hypotheses of Theorem 5.9, but for which no uniform modulus of

continuity for local coordinate mappings and their inverses is possible (over all scales and locations).

Related topics will be discussed in Appendix C.

5.1 Smoothness of Lipschitz and bilipschitz mappings

Another aspect of uniform rectifiability is that it provides the same amount of "smoothness" as when there is a global bilipschitz parameterization. To make this precise, let us first look at the smoothness of Lipschitz and bilipschitz mappings.

A mapping $f : \mathbf{R}^d \to \mathbf{R}^n$ is Lipschitz if there is a constant C so that (5.10) holds for all x, y in $\mathbf{R}^d$. The space of Lipschitz mappings is a bit simpler than the space of bilipschitz mappings, because the former is a vector space (and even a Banach space) while the latter is not. For the purposes of "smoothness" properties, though, there is not really any difference between the two. Bilipschitz mappings are always Lipschitz, and anything that can happen with Lipschitz mappings can also happen with bilipschitz mappings (by adding new components, or considering $x + h(x)$ when $h(x)$ has Lipschitz norm less than 1 to get a bilipschitz mapping).

One should also not worry too much about the difference between Lipschitz mappings which are defined on all of $\mathbf{R}^d$, and ones that are only defined on a subset. Lipschitz mappings into $\mathbf{R}^n$ that are defined on a subset of $\mathbf{R}^d$ can always be extended to Lipschitz mappings on all of $\mathbf{R}^d$. This is a standard fact. There are also extension results for bilipschitz mappings, if one permits oneself to replace the image $\mathbf{R}^n$ with a Euclidean space of larger dimension (which is not too serious in the present context).

For considerations of "smoothness" we might as well restrict our attention to functions which are real-valued, since the $\mathbf{R}^n$-valued case can always be reduced to that.

Two basic facts about Lipschitz functions on $\mathbf{R}^d$ are that they are differentiable almost everywhere (with respect to Lebesgue measure), and that for each $\eta > 0$ they can be modified on sets of Lebesgue measure less than η (depending on the function) in such a way as to become *continuously* differentiable everywhere. See [Fed].

These are well-known results, but they do not tell the whole story. They are not quantitative; they say a lot about the asymptotic behavior (on average) of Lipschitz mappings at very small scales, but they do not say anything about what happens at scales of definite size.

To make this precise, let a Lipschitz mapping $f : \mathbf{R}^d \to \mathbf{R}$ be given, and fix a point $x \in \mathbf{R}^d$ and a radius $t > 0$. We want to measure how well f is approximated by an affine function on the ball $B(x,t)$. To do this we define the quantity $\alpha(x,t)$ by

$$\alpha(x,t) = \inf_{A \in \mathcal{A}} \sup_{y \in B(x,t)} t^{-1} |f(y) - A(y)|. \tag{5.12}$$

Here $\mathcal{A}$ denotes the (vector space) of affine functions on $\mathbf{R}^d$. The part on the right side of (5.12) with just the supremum (and not the infimum) measures how well the particular affine function A approximates f inside $B(x,t)$, and then the infimum gives us the best approximation by any affine function for a particular choice of x and t. The factor of t^{-1} makes $\alpha(x,t)$ scale properly, and be dimensionless. In particular, $\alpha(x,t)$ is uniformly bounded in x and t when f is Lipschitz, because we can take $A(y)$ to be the constant function equal to the value of f at x.

The smallness of $\alpha(x,t)$ provides a manifestation of the smoothness of f. For functions which are twice-continuously differentiable one can get estimates like

$$\alpha(x,t) = O(t), \tag{5.13}$$

using Taylor's theorem. If $0 < \delta < 1$, then estimates like

$$\alpha(x,t) = O(t^\delta) \tag{5.14}$$

(locally uniformly in x) correspond to Hölder continuity of the gradient of f of order δ. Differentiability almost everywhere of f implies that

$$\lim_{t \to 0} \alpha(x,t) = 0 \qquad \text{for almost every } x. \tag{5.15}$$

This does not say anything about any particular t, because one does not know how long one might have to wait before the limiting behavior kicks in.

Here is a simple example. Let us take $d = 1$, and consider the function

$$g_\rho(x) = \rho \cdot \sin(x/\rho) \tag{5.16}$$

on $\mathbf{R}$. Here ρ is any positive number. Now, $g_\rho(x)$ is Lipschitz with norm 1 no matter how ρ is chosen. This is not hard to check; for instance, one can take the derivative to get that

$$g'_\rho(x) = \cos(x/\rho) \tag{5.17}$$

so that $|g'_\rho(x)| \leq 1$ everywhere. This implies that

$$|g_\rho(u) - g_\rho(v)| \leq |u - v| \tag{5.18}$$

for all u and v (and all ρ), because of the mean-value theorem, or the fundamental theorem of calculus. (One also has that $|g'_\rho(x)| = 1$ at some points, so that the Lipschitz norm is always equal to 1.)

If ρ is very small, then one has to wait a long time before the limit in (5.15) takes full effect, because $\alpha(x,t)$ will *not* be small when $t = \rho$. In fact, there is then a positive lower bound for $\alpha(x,t)$ that does not depend on x or ρ (assuming that t is taken to be equal to ρ). This is not hard to verify directly. One does not really have to worry about ρ here, because one can use scaling arguments to reduce the lower bound to the case where $\rho = 1$.

Thus a bound on the Lipschitz norm is not enough to say anything about when the limit in (5.15) will take effect. These examples work uniformly in x, so that one cannot avoid the problem by removing a set of small measure or anything like that.

However, there is something else that one can observe about these examples. Fix a ρ, no matter how large or small. The corresponding quantities $\alpha(x,t)$ will not be too small when t is equal to ρ, as mentioned above, but they will be small when t is either much smaller than ρ, or much larger than ρ. At scales much smaller than ρ, $g_\rho(x)$ is approximately affine, because the smoothness of the sine function has a chance to kick in, while at larger scales $g_\rho(x)$ is simply small outright compared to t (and one can take $A = 0$ as the approximating affine function).

In other words, for the functions $g_\rho(x)$ there is always a bad scale where the $\alpha(x,t)$'s may not be small, and that bad scale can be arbitrarily large or small, but the bad behavior is confined to approximately just one scale.

It turns out that something similar happens for arbitrary Lipschitz functions. The bad behavior cannot always be confined to a single scale — one might have sums of functions like the g_ρ's, but with very different choices of ρ — but, on average, the bad behavior is limited to a bounded number of scales.

Let us be more precise, and define a family of functions which try to count the number of "bad" scales associated to a given point x. Fix a radius $r > 0$, and also a small number ϵ, which will provide our threshold for what is considered "small". We assume that a lipschitz function f on $\mathbf{R}^d$ has been fixed, as before. Given $x \in \mathbf{R}^d$, define $N_r(x)$ to be the number of nonnegative

integers j such that

$$\alpha(x, 2^{-j} r) \geq \epsilon. \tag{5.19}$$

These j's represent the "bad" scales for the point x, and below the radius r.

It is easy to see that (5.15) implies that $N_r(x) < \infty$ for almost all x. There is a more quantitative statement which is true, namely that the average of $N_r(x)$ over any ball B in $\mathbf{R}^d$ of radius r is finite and uniformly bounded, independently of the ball B and the choice of r. That is,

$$r^{-d} \int_B N_r(x) \, dx \leq C(n, \epsilon^{-1} \|f\|_{Lip}), \tag{5.20}$$

where $C(n, s)$ is a constant that depends only on n and s, and $\|f\|_{Lip}$ denotes the Lipschitz norm of f. This is a kind of "Carleson measure condition".

Before we get to the reason for this bound, let us consider some examples. For the functions $g_\rho(x)$ in (5.16), the functions $N_r(x)$ are simply uniformly bounded, independently of x, r, and ρ. This is not hard to check. Notice that the bound does depend on ϵ, i.e., it blows up as $\epsilon \to 0$. As another example, consider the function f defined by

$$f(x) = |x|. \tag{5.21}$$

For this function we have that $N_r(0) = \infty$ as soon as ϵ is small enough. This is because $\alpha(0, t)$ is positive and independent of t, so that (5.19) holds for all j when ϵ is sufficiently small. Thus $N_r(x)$ is not uniformly bounded in this case. In fact it has a logarithmic singularity near 0, with $N_r(x)$ behaving roughly like $\log(r/|x|)$, and this is compatible with (5.20) for B centered at 0. If one is far enough away from the origin (compared to r), then $N_r(x)$ simply vanishes, and there is nothing to do.

In general one can have mixtures of the two types of phenomena. Another interesting class of examples to consider are functions of the form

$$f(x) = \mathrm{dist}(x, F), \tag{5.22}$$

where F is some nonempty subset of $\mathbf{R}^d$ which is not all of $\mathbf{R}^d$, and $\mathrm{dist}(x, F)$ is defined (as usual) by

$$\mathrm{dist}(x, F) = \inf\{|x - z| : z \in F\}. \tag{5.23}$$

It is a standard exercise that such a function f is always Lipschitz with norm at most 1. Depending on the behavior of the set F, this function can have

plenty of sharp corners, like $|x|$ has at the origin, and plenty of oscillations roughly like the ones in the functions g_ρ. In particular, the oscillations can occur at lots of different scales as one moves from point to point. However, one does not really have oscillations at different scales overlapping each other. Whenever the elements of F become dense enough to make a lot of oscillations, the values of f become small in compensation. (One can consider situations where F has points at regularly-spaced intervals, for instance.)

How might one prove an estimate like (5.20)? This is part of a larger story in harmonic analysis, called Littlewood–Paley theory, some of whose classical manifestations are described in [Ste1]. The present discussion is closer in spirit to [Dor] for the measurements of oscillation used, and indeed (5.20) can be derived from the results in [Dor].

There are stronger estimates available than (5.20). Instead of simply counting how often the $\alpha(x,t)$'s are larger than some threshold, as in the definition of $N_r(x)$ above, one can work with sums of the form

$$\Big(\sum_{j=0}^{\infty} \alpha(x, 2^{-j} \, r)^q \Big)^{\frac{1}{q}}. \tag{5.24}$$

The "right" choice of q is 2, but to get this one should modify the definition of $\alpha(x,t)$ (in most dimensions) so that the measurement of approximation of f by an affine function uses a suitable L^p norm, rather than the supremum. (That the choice of $q = 2$ is the "right" one reflects some underlying orthogonality, and is a basic point of Littlewood–Paley theory. At a more practical level, $q = 2$ is best because it works for the estimates for the $\alpha(x,t)$'s and allows reverse estimates for the size of the gradient of f in terms of the sizes of the $\alpha(x,t)$'s.)

In short, harmonic analysis provides a fairly thorough understanding of the sizes of the $\alpha(x,t)$'s and related quantities, and with quantitative estimates. This works for Lipschitz functions, and more generally for functions in Sobolev spaces.

There is more to the matter of smoothness of Lipschitz functions than this, however. The $\alpha(x,t)$'s measure how well a given function f can be approximated by an affine function on a ball $B(x,t)$, but they do not say too much about how these approximating affine functions might change with x and t. In fact, there are classical examples of functions for which the $\alpha(x,t)$'s tend to 0 uniformly as $t \to 0$, and yet the derivative fails to exist at almost every point. Roughly speaking, the affine approximations keep spinning around as $t \to 0$, without settling down on a particular affine function,

as would happen when the derivative exists. (A faster rate of decay for the $\alpha(x,t)$'s, as in (5.14), would prevent this from happening.)

For Lipschitz functions, the existence of the differential almost everywhere implies that for almost every x the gradients of the approximating affine functions on $B(x,t)$ do not have to spin around by more than a finite amount as t goes between some fixed number r and 0. In fact, quantitative estimates are possible, in much the same manner as before. Again one fixes a threshold ϵ, and one can measure how many oscillations of size at least ϵ there are in the gradients of the affine approximations as t ranges between 0 and r. For this there are uniform bounds on the averages of these numbers, just as in (5.20).

This type of quantitative control on the oscillations of the gradients of the affine approximations of f comes from Carleson's Corona construction, as in [Garn]. This construction was initially applied to the behavior of bounded holomorphic functions in the unit disk of the complex plane, but in fact it is a very robust real-variable method. For example, the type of bound just mentioned in the previous paragraph (on the average number of oscillations of the gradients of the affine approximations as t goes from r to 0) is completely analogous to one for the boundary behavior of harmonic functions given in Corollary 6.2 on p348 of [Garn].

A more detailed discussion of the Corona construction in the context of Lipschitz functions can be found in Chapter IV.2 of [DaviS5].

The Corona construction and the known estimates for affine approximations as discussed above provide a fairly complete picture of the "smoothness" of Lipschitz functions. They also provide an interesting way to look at "complexity" of Lipschitz functions, and one that is quite different from what is suggested more naively by the definition (5.10).

5.2 Smoothness and uniform rectifiability

The preceding discussion of smoothness for Lipschitz and bilipschitz mappings has natural extensions to the geometry of sets in Euclidean spaces. Instead of approximations of functions by affine functions, one can consider approximations of sets by affine planes. Differentials of mappings correspond to tangent planes for sets.

One can think of "embedding" the discussion for functions into one for sets by taking a function and replacing it with its graph. This is consistent with the correspondence between affine functions and d-planes, and between

differentials and tangent planes.

How might one generalize the $\alpha(x,t)$'s (5.12) to the context of sets? Fix a set E in $\mathbf{R}^n$ and a dimension $d < n$, and let $x \in E$ and $t > 0$ be given. In analogy with (5.12), consider the quantity $\beta(x,t)$ defined by

$$\beta(x,t) = \inf_{P \in \mathcal{P}_d} \sup\{t^{-1} \operatorname{dist}(y,P) : y \in E \cap B(x,t)\}. \tag{5.25}$$

Here $\mathcal{P}_d$ denotes the set of d-dimensional affine planes in $\mathbf{R}^n$, and $\operatorname{dist}(y,P)$ is defined as in (5.23). In other words, we take a "snapshot" of E inside the ball $B(x,t)$, and we look at the optimal degree of approximation of E by d-planes in $B(x,t)$. The factor of t^{-1} in (5.25) makes $\beta(x,t)$ a scale-invariant, dimensionless quantity. Notice that $\beta(x,t)$ is always less than or equal to 1, no matter the behavior of E, as one can see by taking P to be any d-plane that goes through x. The smoothness of E is reflected in how small $\beta(x,t)$ is.

If E is the image of a bilipschitz mapping $\phi : \mathbf{R}^d \to \mathbf{R}^n$, then there is a simple correspondence between the $\beta(x,t)$'s on E and the $\alpha(z,s)$'s for ϕ on $\mathbf{R}^d$. This permits one to transfer the estimates for the α's on $\mathbf{R}^d$ to estimates for the β's on E, and one could also go backwards.

It turns out that the type of estimates that one gets for the β's in this way when E is bilipschitz equivalent to $\mathbf{R}^d$ also work when E is uniformly rectifiable. Roughly speaking, this because the estimates for the α's and β's are not uniform ones, but involve some kind of integration, and in a way which is compatible with the measure-theoretic aspects of the Definition 5.4. This is very much analogous to results in the context of BMO functions, especially a theorem of Strömberg. (See Chapter IV.1 in [DaviS5] for some general statements of this nature.)

To my knowledge, the first person to look at estimates like these for sets was P. Jones [Jon1]. In particular, he used the sharp quadratic estimates that correspond to Littlewood–Paley theory to give a new approach to the L^2 boundedness of the Cauchy integral operator on nonsmooth curves. Here "quadratic" means $q = 2$ in the context of (5.24).

In [Jon3], Jones showed how quadratic estimates on the β's could actually be used to *characterize* subsets of rectifiable curves. The quadratic nature of the estimates, which come naturally from orthogonality considerations in Littlewood–Paley theory and harmonic analysis, can, in this context, be more directly linked to the ordinary Pythagorean theorem, as in [Jon3]. A completion of Jones' results for 1-dimensional sets in Euclidean spaces of

higher dimension was given in [Oki].

Analogues of Jones' results for (Ahlfors-regular) sets of higher dimension are given in [DaviS3]. More precisely, if E is a d-dimensional Ahlfors-regular set in $\mathbf{R}^n$, then the uniform rectifiability of E is equivalent to certain quadratic Carleson measure conditions for quantities like $\beta(x,t)$ in (5.25). One cannot use $\beta(x,t)$ itself in general, with the supremum on the right side of (5.25) (there are counterexamples due to Fang and Jones), but instead one can replace the supremum with a suitable L^p norm for a range of p's that depends on the dimension (and is connected to Sobolev embeddings). This corresponds to the situation for sharp estimates of quantities like $\alpha(x,t)$ in the context of Lipschitz functions, as in [Dor].

The problem of building parameterizations is quite different when $d > 1$ than in the 1-dimensional case. This is a basic fact, and a recurring theme of classical topology. Making parameterizations for 1-dimensional sets is largely a matter of ordering, i.e., lining up the points in a good way. For rectifiable curves there is a canonical way to regulate the "speed" of a parameterization, using arclength. In higher dimensions none of these things are true, although conformal coordinates sometimes provide a partial substitute when $d = 2$. See [DaviS4, MülŠ, HeiKo1, Sem3, Sem7]. In [DeTY] a different kind of "normalized coordinates" are discussed for $d = 3$, but the underlying partial differential equation is unfortunately not elliptic. Part of the point of uniform rectifiability was exactly to try to come to grips with the issue of parameterizations in higher dimensions. (See also Appendix C in connection with these topics.)

Although this definition (5.25) of $\beta(x,t)$ provides a natural version of the $\alpha(x,t)$'s from (5.12), it is not the only choice to consider. There is a "bilateral" version, in which one measures both the distance from points in E to the approximating d-plane (as in (5.25)) as well as distances from points in the d-plane to E. Specifically, given a set E in $\mathbf{R}^n$, a point $x \in E$, a radius $t > 0$, and a d-plane P in $\mathbf{R}^n$, set

$$\begin{aligned} \text{(5.26)}\quad \text{Approx}(E,P,x,t) &= \sup\{t^{-1}\,\text{dist}(y,P) : y \in E \cap B(x,t)\} \\ &\quad + \sup\{t^{-1}\,\text{dist}(z,E) : z \in P \cap B(x,t)\} \end{aligned}$$

and then define the bilateral version of $\beta(x,t)$ by

$$\text{(5.27)}\qquad b\beta(x,t) = \inf_{P \in \mathcal{P}_d} \text{Approx}(E,P,x,t).$$

This takes "holes" in E into account, which the definition of $\beta(x,t)$ does not. For instance, $\beta(x,t) = 0$ if and only if there is a d-plane P_0 such that every

point in $E \cap B(x,t)$ lies in P_0, while for $b\beta(x,t)$ to be 0 it should also be true that every point in $P_0 \cap B(x,t)$ lies in E (assuming that E is closed, as in the definition of Ahlfors regularity).

It turns out that the $b\beta(x,t)$'s behave a bit differently from the $\beta(x,t)$'s, in the following sense. Imagine that we do not look for something like sharp quadratic estimates, as we did before, but settle for cruder "thresholding" conditions, as discussed in Subsection 5.1. In other words, one might fix an $\epsilon > 0$, and define a function $N_r'(x)$ which counts the number of times that $b\beta(x, 2^{-j}\, r)$ is greater than or equal to ϵ, with $j \in \mathbf{Z}_+$ (as in the discussion around (5.19)). For uniformly rectifiable sets one has bounds on the averages of $N_r'(x)$ exactly as in (5.20), but now integrating over E instead of $\mathbf{R}^d$. A slightly surprising fact is that the converse is also true, i.e., estimates like these for the $b\beta$'s are sufficient to imply the uniform rectifiability of the set E, at least if E is Ahlfors-regular of dimension d. This was proved in [DaviS5].

In the context of functions, this type of thresholding condition is too weak, in that one can have the $\alpha(x,t)$'s going to 0 uniformly as $t \to 0$ for functions which are differentiable almost nowhere, as mentioned in Subsection 5.1. Similarly, there are Ahlfors-regular sets which are "totally unrectifiable" (in the sense of [Fal, Fed, Mat]) and have the $\beta(x,t)$'s tending to 0 uniformly as $t \to 0$. (See [DaviS3].) For the $b\beta$'s the story is simply different. On the other hand, the fact that suitable thresholding conditions on the $b\beta$'s are sufficient to imply uniform rectifiability relies heavily on the assumption that E be Ahlfors-regular, while mass bounds are part of the conclusion (rather than the hypothesis) in Jones' results, and no counterpart to the mass bounds are included in the above-mentioned examples for functions. One does have mass bounds for the examples in [DaviS3] (of totally-unrectifiable Ahlfors-regular sets for which the $\beta(x,t)$'s tend to 0 uniformly as $t \to 0$), and there the issue is more in the size of the holes in the set. The $b\beta$'s, by definition, control the sizes of holes. Note that this result for the $b\beta$'s does have antecedents for the classical notion of (countable) rectifiability, as in [Mat].

There are a number of variants of the $b\beta$'s, in which one makes comparisons with other collections of sets besides d-planes, like unions of d-planes, for instance. See [DaviS5].

Perhaps the strongest formulation of smoothness for uniformly rectifiable sets is the existence of a "Corona decomposition". This is a geometric version of the information that one can get about a Lipschitz function from the methods of Carleson's Corona construction (as mentioned in Subsection 5.1). Roughly speaking, in this condition one controls not only how often E is well-

approximated by a d-plane, but how fast the d-planes turn as well. This can also be formulated in terms of good approximations of E by flat Lipschitz graphs.

Although a bit technical, the existence of a Corona decomposition is perhaps the most useful way of managing the complexity of a uniformly rectifiable set. Once one has a Corona decomposition, it is generally pretty easy to derive whatever else one would like to know. Conversely, in practice the existence of a Corona decomposition can be a good place to start if one wants to prove that a set is uniformly rectifiable.

In fact, there is a general procedure for finding a Corona decomposition when it exists, and one which is fairly simple (and very similar to Carleson's Corona construction). The difficult part is to show that this procedure works in the right way, with the correct estimates. Specifically, it is a stopping-time argument, and one does not want to have to stop too often. This is a nice point, because in general it is not so easy to build something like a good parameterization of a set, even if one knows a priori that it exists. In this context, there are in principle methods for doing this.

See [DaviS2, DaviS3, DaviS5, DaviS10, Sem1] for more information about Corona decompositions of uniformly rectifiable sets and the way that they can be used. The paper [DaviS10] is written in such a way as to try to convey some of the basic concepts and constructions without worrying about why the theorems are true (which is much more complicated). In particular, the basic procedure for finding Corona decompositions when they exist is discussed. See [GarnJ, Jon1, Jon3] for some other situations in which Carleson's Corona construction is used geometrically.

5.3 A class of variational problems

Uniform rectifiability is a pretty robust condition. If one has a set which looks roughly as though it ought to be uniformly rectifiable, then there is a good chance that it is. This as opposed to sets which look roughly as though they should admit a well-behaved (homeomorphic) parameterization, and do not (as discussed before).

In this subsection we would like to briefly mention a result of this type, concerning a minimal surface problem with nonsmooth coefficients. Let $g(x)$ be a Borel measurable function on $\mathbf{R}^n$, and assume that g is positive,

bounded, and bounded away from 0, so that

$$0 < m \le g(x) \le M \tag{5.28}$$

for some constants m, M and all $x \in \mathbf{R}^n$. Let Q_0, Q_1 be a pair of (closed) cubes in $\mathbf{R}^n$, with sides parallel to the axes, and assume that Q_0 is contained in the interior of Q_1.

Let U be an open subset of Q_1 which contains the interior of Q_0. Consider an integral like

$$\int_{\partial U} g(x) \, d\nu_U(x), \tag{5.29}$$

where $d\nu_U$ denotes the measure that describes the $(n-1)$-dimensional volume of subsets of ∂U. This would be defined as in calculus when ∂U is at least a little bit smooth (like C^1), but in general one has to be more careful. One can simply take for $d\nu_U$ the restriction of $(n-1)$-dimensional Hausdorff measure to ∂U, but for technical reasons it is often better to define $d\nu_U$ using distributional derivatives of the characteristic function of U, as in [Giu]. For this one would work with sets U which have "finite perimeter", which means exactly that the distributional first derivatives of the characteristic function of U are measures of finite mass.

Here is one way in which this kind of functional, and the minimization of this kind of functional, can come up. Let F be a closed subset of $\mathbf{R}^n$. Imagine that one is particularly interested in domains U which have their boundary contained in F, or very nearly so. On the other hand, one might also wish to limit irregularities in the behavior of the boundary of U. For this type of situation one could choose g so that it is much smaller on F than on the complement of F, and then look for minimizers of (5.29) to find domains with a good balance between the behavior of ∂U and the desire to have it be contained (as much as possible) in F. (See [DaviS9] for an example of this.)

When do minimizers of (5.29) exist, and how do they behave? If one works with sets of finite perimeter, and if the function g is lower semi-continuous, then one can obtain the existence of minimizers through standard techniques (as in [Giu]). That is, one takes limits of minimizing sequences for (5.29) using weak compactness, and one uses the lower semi-continuity of g to get lower semi-continuity of (5.29) with respect to suitable convergence of the U's. The latter ensures that the limit of the minimizing sequence is actually a minimum. Note that the "obstacle" conditions that U contain the interior of Q_0 and be contained in Q_1 prevents the minimization from collapsing into something trivial.

As to the behavior of minimizers of (5.29), one cannot expect much in the way of smoothness in general. For instance, if the boundary of U can be represented locally as the graph of a Lipschitz function, then U in fact minimizes (5.29) for a suitable choice of g. Specifically, one can take g to be a sufficiently small positive constant on ∂U, and to be equal to 1 everywhere else. That such a choice of g works is not very hard to establish, and more precise results are given in [DaviS9].

Conversely, minimizers of (5.29) are always Ahlfors-regular sets of dimension $n-1$, and uniformly rectifiable. This is shown in [DaviS9], along with some additional geometric information which is sufficient to characterize the class of sets U which occur as minimizers for functionals of the form (5.29) (with g bounded and bounded away from 0). If a set U arises as the minimizer for some g, it is also a minimizer with g chosen as above, i.e., a small positive constant on ∂U and equal to 1 everywhere else.

The same regularity results work for a suitable class of "quasiminimizers" of the usual area functional, and one that includes minimizers for (5.29) as a special case.

Uniform rectifiability provides a natural level of structure for situations like this, where stronger forms of smoothness cannot be expected, but quantitative bounds are reasonable to seek. Note that properties of ordinary rectifiability always hold for boundaries of sets of finite perimeter, regardless of any minimizing or quasiminimizing properties. See [Giu].

Analogous results about regularity work for sets of higher codimension as well, although this case is more complicated technically. See [DaviS11] for more information. One can use this framework of minimization (with respect to nonsmooth coefficient functions g) as a tool for studying the structure of sets in $\mathbf{R}^n$ with upper bounds on their d-dimensional Hausdorff measure and lower bounds for their d-dimensional topology. This brings one back to Theorem 5.9 and related questions, and in particular more "localized" versions of it.

To put it another way, minimization of functionals like these can provide useful means for obtaining "existence results" for approximate parameterizations with good behavior, through uniform rectifiability. See [DaviS9, DaviS11]. Part of the motivation for this came from an earlier argument of Morel and Solimini [MoreS]. Their argument concerned the existence of curves containing a given set, with good properties in terms of the distribution of the arc-length measure of these curves, under more localized conditions on the given set (at all locations and scales). See Lemma 16.27 on p207

of [MoreS].

A Fourier transform calculations

If $\phi(x)$ is an integrable function on $\mathbf{R}^n$, then its *Fourier transform* $\widehat{\phi}(\xi)$ is defined (for $\xi \in \mathbf{R}^n$) by

$$\widehat{\phi}(\xi) = \int_{\mathbf{R}^n} e^{i\,\langle x,\xi\rangle}\,\phi(x)\,dx. \tag{A.1}$$

Here $\langle x, \xi\rangle$ denotes the usual inner product for $x, \xi \in \mathbf{R}^n$, and $i = \sqrt{-1}$. Often one makes slightly different conventions for this definition — with some extra factors of π around, for instance — but we shall not bother with this.

A key feature of the Fourier transform is that it diagonalizes differential operators. Specifically, if ∂_k denotes the operator $\partial/\partial x_k$ on $\mathbf{R}^n$, then

$$\widehat{(\partial_k \phi)}\,(\xi) = i\,\xi_k\,\widehat{\phi}(\xi), \tag{A.2}$$

i.e., differentiation is converted into mere multiplication. For this one should either make some differentiability assumptions on ϕ, so that the left side can be defined in particular, or one should interpret this equation in the sense of tempered distributions on $\mathbf{R}^n$. The Fourier transform also carries out this diagonalization in a controlled manner. That is, there is an explicit inversion formula (which looks a lot like the Fourier transform itself), and the Fourier transform preserves the L^2 norm of the function ϕ, except for a multiplicative constant, by the Plancherel theorem. See [SteW] for these and other basic facts about the Fourier transform.

Using Plancherel's theorem, it is very easy to give another proof of the L^2 estimate (2.2) from Section 2, and to derive many other inequalities of a similar nature. One can also use the Fourier transform to give a precise definition of the operator $R = \partial_j \partial_k / \Delta$, where Δ is the Laplace operator $\sum_{\ell=1}^n \partial_\ell^2$. Specifically, one can define it through the equation

$$\widehat{(R\phi)}\,(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}\,\widehat{\phi}(\xi). \tag{A.3}$$

If $m(\xi)$ is any bounded function on $\mathbf{R}^n$, then

$$\widehat{(T\phi)}\,(\xi) = m(\xi)\,\widehat{\phi}(\xi) \tag{A.4}$$

defines a bounded operator on $L^2(\mathbf{R}^n)$. In general these operators are not bounded on L^p for any other value of p, but this is true for many of the operators that arise naturally in analysis. For instance, suppose that $m(\xi)$ is homogeneous of degree 0, so that

$$m(t\xi) = m(\xi) \qquad \text{when } t > 0, \tag{A.5}$$

and that $m(\xi)$ is smooth away from the origin. Then the associated operator T is bounded on L^p for all p with $1 < p < \infty$. See [Ste1, SteW]. Note that this criterion applies to the specific choice of $m(\xi)$ in (A.3) above.

For a multiplier operator as in (A.4) to be bounded on L^1 or L^∞ is even more exceptional than for L^p boundedness when $1 < p < \infty$. (See [SteW].) For instance, if m is homogeneous, as above, and not constant, then the corresponding operator cannot be bounded on L^1 or L^∞. However, if m is homogeneous and smooth away from the origin, then the operator T in (A.4) does determine a bounded operator from L^∞ into BMO. See [GarcR, Garn, Jou, Ste2]. In fact, T determines a bounded operator from BMO to itself.

Here is another example. Let ϕ now be a mapping from $\mathbf{R}^2$ to itself, with components ϕ_1, ϕ_2. Consider the differential $d\phi$ of ϕ as a matrix-valued function, namely,

$$\begin{pmatrix} \partial_1\phi_1 & \partial_1\phi_2 \\ \partial_2\phi_1 & \partial_2\phi_2 \end{pmatrix}. \tag{A.6}$$

(Let us assume that ϕ is smooth enough that the differential is at least some kind of function when taken in the sense of distributions, although one can perfectly well think of $d\phi$ as a matrix-valued distribution.) Let A and S denote the antisymmetric and symmetric parts of $d\phi$, respectively, so that

$$A = \frac{d\phi - d\phi^t}{2}, \qquad S = \frac{d\phi + d\phi^t}{2}, \tag{A.7}$$

where $d\phi^t$ denotes the transpose of $d\phi$.

In this case of 2×2 matrices, the antisymmetric part A really contains only one piece of information, namely

$$\partial_1\phi_2 - \partial_2\phi_1. \tag{A.8}$$

It is not hard to check that this function can be reconstructed from the entries of S through operators of the form (A.4), using functions m which

are homogeneous of degree 0 and smooth away from the origin. For this one should add some mild conditions on ϕ, like compact support, to avoid the possibility that S vanishes identically but A does not.

Under these conditions, we conclude that the L^p norm of A is always bounded by a constant multiple of the L^p norm of S, $1 < p < \infty$, and that the BMO norm of A is controlled by the L^∞ (or BMO) norm of S. (For the case of BMO norms, the possibility that S vanishes but A does not causes no trouble, because A will be constant in that case.)

This example is really a "linearized" version of the problem discussed in Section 1. Specifically, let us think of $f : \mathbf{R}^2 \to \mathbf{R}^2$ as being of the form

$$f(x) = x + \epsilon\, \phi(x), \tag{A.9}$$

where ϵ is a small parameter. The extent to which f distorts distances is governed by the matrix-valued function $df^t\, df$, which we can write out as

$$df^t\, df = I + 4\,\epsilon\, S + \epsilon^2\, d\phi^t d\phi. \tag{A.10}$$

Thus the linear term in ϵ is governed by S, while A controls the leading behavior in ϵ of the "rotational" part of df.

B Mappings with branching

In general, there can be a lot of trouble with existence and complexity of homeomorphisms (with particular properties, like specified domain and range). If one allows mappings with *branching*, then the story can be very different.

As a basic example of this, there is a classical result originating with Alexander to the effect that any oriented pseudomanifold of dimension n admits an orientation-preserving branched covering over the n-sphere. Let us state this more carefully, and then see how it is proved.

Let M be a finite polyhedron. We assume that M is given as a finite union of n-dimensional simplices that meet only in their boundary faces (so that M is really a simplicial complex). To be a pseudomanifold means that every $(n-1)$-dimensional face in M arises as the boundary face of exactly *two* n-dimensional simplices. In effect this says that M looks like a manifold away from its codimension-2 skeleton (the corresponding statement for the codimension-1 skeleton being automatic). For the present purposes it would

be enough to ask that every $(n-1)$-dimensional face in M arise as the boundary face of *at most* two n-dimensional simplices, which would be like a "pseudomanifold with boundary".

An *orientation* for an n-dimensional pseudomanifold M means a choice of orientation (in the usual sense) for each of the constituent n-dimensional simplices in M, with compatibility of orientations of adjacent n-dimensional simplices along the common $(n-1)$-dimensional face. In terms of algebraic topology, this means that the sum of the n-dimensional simplices in M, with their orientations, defines an n-dimensional *cycle* on M.

For the purposes of the Alexander-type result, it will be convenient to think of the n-sphere as consisting of two standard simplices S_1 and S_2 glued together along the boundary. This is not quite a polyhedron in the usual (affine) sense, but one could easily repair this by subdividing S_1 or S_2. We also assume that S_1 and S_2 have been oriented, and have opposite orientations relative to their common boundary.

To define a mapping from M to the n-sphere one would like to simply identify each of the constituent n-dimensional simplices in M with S_1 or S_2 in a suitable manner. Unfortunately, this does not work, even when $n = 2$, but the problem can be fixed using a *barycentric subdivision* of M. Recall that the *barycenter* of a simplex (embedded in some vector space) is the point in the interior of the simplex which is the average of the vertices of the simplex. The *set of barycenters* for M means the set of barycenters of all of the constituent simplices in M (viewed as a simplicial complex), of all dimensions, including 0. In particular, the set of barycenters for M includes the vertices of M (which are themselves 0-dimensional simplices, and their own barycenters). The barycentric subdivision of M is a refinement of M as a simplicial complex whose vertices are exactly the set of barycenters of M. In other words, the set M as a whole does not change, just its decomposition into simplices, which is replaced by a finer decomposition.

Here is a precise description of the simplices in the barycentric subdivision of M. Let $s_0, s_1, \ldots, s_k$ be a finite sequence of simplices in M, with each s_i an i-dimensional simplex which is a face of s_{i+1} (when $i < k$). Let $b(s_i)$ denote the barycenter of s_i. Then $b(s_0), b(s_1), \ldots, b(s_k)$ are affinely independent, and hence determine a k-dimensional simplex. The simplices that arise in this manner are precisely the ones used for the barycentric subdivision of M. (See p123 of [Spa] for more details.)

Let $\widetilde{V}$ denote the set of all vertices in the barycentric subdivision of M. This is the same as the set of points which arise as barycenters of simplices

in the original version of M, and in particular we have a natural mapping from $\widetilde{V}$ to the integers $\{0, 1, \ldots, n\}$, defined by associating to each point b in $\widetilde{V}$ the dimension of the simplex from which it was derived.

If T is a k-dimensional simplex in the barycentric subdivision of M, then the mapping from $\widetilde{V}$ to $\{0, 1, \ldots, n\}$ just described induces a one-to-one correspondence between the $k + 1$ vertices of T and the set $\{0, 1, \ldots, k\}$. This follows easily from the definitions.

We are now ready to define our mapping from M to the n-sphere. There are exactly $n + 1$ vertices in our realization of the n-sphere as the gluing of S_1 and S_2. Let us identify these vertices with the integers from 0 to n. Thus our mapping from $\widetilde{V}$ to $\{0, 1, \ldots, n\}$ can now be interpreted as a mapping from the vertices of the barycentric subdivision of M to the vertices of the n-sphere.

This mapping between vertices admits a canonical linear extension to each k-dimensional simplex, $k < n$, in the barycentric subdivision of M. For the n-dimensional simplices the extension is uniquely determined once one chooses S_1 or S_2 for the image of the simplex. Because of the orientations, there is only one natural choice of S_1 or S_2 for each n-dimensional simplex T, namely the one so that the linear mapping from T onto S_j is orientation-preserving.

In the end we get a mapping from the barycentric subdivision of M to the n-sphere which preserves orientations and which defines an affine isomorphism from each n-dimensional simplex T in the domain onto one of S_1 and S_2. This uses the fact that our initial mapping between vertices was always one-to-one on the set of vertices in any given simplex in the domain, by construction.

This completes the proof. We should emphasize that the singularities of the mapping from M to the n-sphere — i.e., the places where it fails to be a local homeomorphism — are confined to the codimension-2 skeleton of the barycentric subdivision of M. This is because of the orientation and pseudomanifold conditions, which ensure that if a point in M lies in the interior of an $(n - 1)$-dimensional simplex in the barycentric subdivision of M, then the (two) adjacent n-simplices at that point are not sent to the same S_j in the image.

The idea of branching also makes sense for mappings that are not piecewise-linear, and there are well-developed notions of "controlled geometry" in this case, as with the classes of quasiregular mappings and mappings of bounded length distortion. See [HeiKiM, MartRiV1, MartRiV2, MartRiV3, MartV,

Res, Ric1, Väi2, Vuo], for instance. In [HeiR1, HeiR2] there are examples where branching maps of controlled geometry can be constructed but suitable homeomorphisms either do not exist or must distort distances more severely.

Sullivan [Sul2, Sul3] has proposed some mechanisms by which the existence of local (controlled) branching maps can be deduced, and some ideas for studying obstructions to controlled homeomorphic coordinates.

See [Gut+, HeiKi, MartRyV] for some recent results about branching and regularity conditions under which it does not occur. A broader and more detailed discussion of mappings with branching can be found in [HeiR2]. For some real-variable considerations of mappings which may branch but enjoy substantial geometric properties, see [Davi4, Jon2, DaviS4].

C More on existence and behavior of homeomorphisms

C.1 Wildness and tameness phenomena

Consider the following question. Let n be a positive integer, and let K be a compact subset of $\mathbf{R}^n$. If K is homeomorphic to the unit interval $[0, 1]$, is there

(C.1) a global homeomorphism from $\mathbf{R}^n$ onto itself which maps K to a straight line segment?

If $n = 1$, then K itself is a closed line segment, and the answer is "yes". When $n = 2$, the answer is also "yes", but this is more complicated, and is more in the spirit of the Schönflies theorem in the plane. See [Moi], especially Chapter 10.

When $n \geq 3$, the answer to the question above can be "no". An arc K is said to be "tame" (or flat) when a homeomorphism does exist as in (C.1), and "wild" when it does not exist. See [Moi] for some examples of wild arcs in $\mathbf{R}^3$.

Smooth arcs are always tame, as are polygonal arcs, i.e., arcs made up of *finitely* many straight line segments. For these one can take the corresponding homeomorphism to be smooth or piecewise-linear as well. (Compare with Theorem 1 on p134 of [Moi], for instance.) In order for an arc to be wild, some amount of infinite processes are needed.

A simple closed curve in $\mathbf{R}^3$ might be smooth or polygonal and still *knotted*, so that there does not exist a homeomorphism of $\mathbf{R}^3$ onto itself which maps the curve onto a standard circle (inside a standard 2-dimensional plane in $\mathbf{R}^3$). There are many well-known examples of this, like the trefoil knot. Thus, for a closed curve, one defines "wildness" in a slightly differently way, in terms of the existence of local flattenings, for instance. This turns out to be compatible with the case of arcs (for which there is no issue of knottedness), and there are some other natural variants of this.

Here is another basic example, for sets of higher dimension. Suppose that γ is a simple closed curve in $\mathbf{R}^3$, which is a polygonal curve, and which represents the trefoil knot. Consider the cone over γ, which gives a 2-dimensional polyhedron in $\mathbf{R}^4$, and which is in fact piecewise-linearly equivalent to a standard 2-dimensional cell. One can show that this embedding of the 2-cell is not locally flat at the cone point, i.e., it cannot be straightened out to agree with a standard (geometrically flat) embedding by a homeomorphism defined on a neighborhood in $\mathbf{R}^4$ of the cone point. Similar phenomena occur for codimension-2 embeddings in $\mathbf{R}^n$ for all $n \geq 4$, as in Example 2.3.2 on p59-60 of [Rus1].

This phenomenon is special to codimension 2, however. Specifically, a piecewise-linear embedding of a k-dimensional piecewise-linear manifold into $\mathbf{R}^n$ is locally topologically flat if $n - k \neq 2$ (or if $k = 1$ and $n = 3$, as before). See Theorem 1.7.2 on p34 of [Rus1].

In the context of piecewise-linear embeddings, one can also look for local flattenings which are piecewise-linear. A similar remark applies to other categories of mappings. We shall not pursue this here.

Wild embeddings of cells and spheres (and other manifolds) exist in $\mathbf{R}^n$ for all $n \geq 3$, and for all dimensions of the cells and spheres (from 1 to $n-1$). This includes embeddings of cells and spheres which are not equivalent to piecewise-linear embeddings in codimension 2. We shall mostly consider here issues of existence of topological flattenings or local flattenings, and embeddings which are not normally given as piecewise-linear.

See [Bin6, Bur, BurC, Can1, Dave1, Dave2, Edw1, Moi, Rus1, Rus2] for more information, and for further references. Let us also mention that embeddings, although wild, may still enjoy substantial good behavior. For instance, they may be bilipschitz, as in (4.7), or quasisymmetric, in the sense of [TukV]. (Roughly speaking, an embedding is *quasisymmetric* if relative distances are approximately preserved, rather than distances themselves, as for a bilipschitz mapping.) See [Geh1, LuuV, Väi3] for some basic results

about this.

As another version of wildness for embeddings, imagine that one has a compact set C in some $\mathbf{R}^n$, and that C is homeomorphic to the usual middle-thirds Cantor set. Can one move C to a subset of a straight line in $\mathbf{R}^n$, through a homeomorphism from $\mathbf{R}^n$ onto itself?

When $n = 1$ this is automatically true. It is also true when $n = 2$; see [Moi], especially Chapter 13. In higher dimensions it is not true in general, as is shown by a famous construction of Antoine ("Antoine's necklaces"). See Chapter 18 of [Moi] and [Bla].

How can one tell when a set is embedded wildly or not? As a simple case, let us consider Cantor sets. If C is a compact subset of $\mathbf{R}^n$ which lies in a line and is homeomorphic to the Cantor set, and if n is at least 3, then the complement of C in $\mathbf{R}^n$ is simply-connected. This is not hard to see. Basically, if one takes a loop in the complement of C and fills it with a disk in $\mathbf{R}^n$, and if that disk happens to run into C, then one can make small perturbations of the disk to avoid intersecting C.

The complement of C is also simply-connected if there is a global homeomorphism from $\mathbf{R}^n$ onto itself which maps C into a line. This is merely because the homeomorphism itself permits one to reduce to the previous case.

However, Antoine's necklaces have the property that their complements are *not* simply-connected. See [Moi, Bla]. Note that the *homology* of the complement of a compact set in $\mathbf{R}^n$ is controlled through the intrinsic topology of the set itself, as in Alexander duality [Spa]. In particular, while the complement of an Antoine's necklace may not be simply-connected, its 1-dimensional homology does vanish.

Versions of the fundamental group play an important role for wildness and taming in general, and not just for Cantor sets. For this one may not take (or want to take) the fundamental group of the whole complementary set, but look at more localized forms of the fundamental group. A specific and basic version of this is the following. Suppose that F is a closed set inside of some $\mathbf{R}^n$. Given a point $p \in F$, and a loop γ in $\mathbf{R}^n \backslash F$ which lies close to p, one would like to know whether it is possible to contract γ to a point in $\mathbf{R}^n \backslash F$ while staying in a small neighborhood of p. This second neighborhood of p might not be quite as small the first one; a precise statement would say that for every $\epsilon > 0$ there is a $\delta > 0$ so that if γ lies in $B(p, \delta) \cap (\mathbf{R}^n \backslash F)$, then γ can be contracted to a point in $B(p, \epsilon) \cap (\mathbf{R}^n \backslash F)$.

This type of condition is satisfied by standard embeddings of sets into

$\mathbf{R}^n$, like Cantor sets, cells, and spheres, at least when the dimension of the set is different from $n-2$. For a point in $\mathbf{R}^2$, or a line segment in $\mathbf{R}^3$, etc., one would get $\mathbf{Z}$ for the corresponding localized fundamental group of the complement of the set. (In the case of a line segment in $\mathbf{R}^3$, one should restrict one's attention to points p in the interior of the segment for this.)

Conversely, there are results which permit one to go backwards, and say that localized fundamental group conditions for the complement like these (localized simple-connectedness conditions in particular) lead to tameness of a given set, or other kind of "standard" (non-wild) behavior. See [Bin5, Bin6, Bin8, Bur, BurC, Can1, Can2, Dave1, Dave2, Edw1, Moi, Qui1, Qui2, Rus1, Rus2] for more information about localized fundamental groups and their role in wildness phenomena and taming theorems (and for related matters and further references).

Fundamental groups and localized versions of them have a basic role in geometric topology in general. Some aspects of this came up before in Section 3, and we shall encounter some more in this appendix.

One might wonder why π_1 and localized versions of it play such an important role. Some basic points behind this are as follows. For homology (or cohomology), one often has good information from data in the given situation through standard results in algebraic topology, like duality theorems. This definitely does not work for π_1, as shown by many examples, including the ones mentioned above, and others later in this appendix. In circumstances with suitable simple-connectivity, one can pass from information about homology to information about homotopy (in general dimensions), as in the Hurewicz and Whitehead theorems. See [Bred1, Spa]. For many constructions, homotopy is closer to what one really needs.

Another basic point concerns the effect of *stabilization*. A wild embedding of a set into some $\mathbf{R}^n$ can become tame when viewed as an embedding into an $\mathbf{R}^m$ with $m > n$ ($m = n+1$ in particular). The same is true for knotting. For instance, a smooth loop may be knotted in $\mathbf{R}^3$, but when viewed as a subset of $\mathbf{R}^4$, it is always unknotted. This is easy to see in explicit examples (like a trefoil knot).

For some simple and general results about wild embeddings in $\mathbf{R}^n$ becoming tame in a larger $\mathbf{R}^m$, see Proposition 4 on p84 of [Dave2], and the corollaries on p85 of [Dave2]. These involve a famous device of Klee. In concrete examples, one can often see the taming in a larger-dimensional space directly, and explicitly. Examples of wild sets are often made with the help of various linkings, or something like that, and in a higher-dimensional space

one can disentangle the linked parts. This can be accomplished by taking individual pieces and pulling them into a new dimension, moving them around freely there, and then putting them back into the original $\mathbf{R}^n$ in a different way.

This simplifying effect of stabilization also fits with the role of localized versions of fundamental groups indicated before. Let F be a closed set inside of some $\mathbf{R}^n$, and imagine that one has a loop γ in $\mathbf{R}^n \backslash F$ which lies in a small ball centered at a point in F. In the condition that was discussed earlier, one would like to contract γ to a point in the complement of F, while remaining in a small ball. If one thinks of γ and F as being also inside $\mathbf{R}^{n+1}$, then it is easy to contract γ to a point in the complement of F in $\mathbf{R}^{n+1}$, while remaining in a small ball. Specifically, one can first translate γ into a parallel copy of $\mathbf{R}^n$ inside of $\mathbf{R}^{n+1}$, i.e., into $\mathbf{R}^n \times \{a\}$ for some $a \neq 0$ rather than $\mathbf{R}^n \times \{0\}$ in $\mathbf{R}^{n+1}$ (using the obvious identifications). This parallel copy is then disjoint from F, and one can contract the loop in a standard way. Note that this argument works independently of the behavior of F.

Using taming theorems based on localized fundamental group conditions, and considerations like those in the previous paragraph, one can get stronger results on tameness that occurs from stabilization than the ones on p84-85 in [Dave2] mentioned above. More precisely, instead of needing k extra dimensions in some cases, it is enough to go from $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$. Compare with the bottom of p390 and the top of p391 in [Dave1], and the references indicated there. (Compare also with the remarks on p452 of [Can1].)

At any rate, this type of phenomenon, of objects becoming more "tame" or simple after stabilization, is a very basic one in geometric topology (as well as other areas, for that matter). We shall encounter a number of other instances of this in this appendix. As in the present setting, the effect of stabilization is also frequently related to conditions concerning localized fundamental groups. I.e., such conditions often become true, and in a simple way, after stabilization.

C.2 Contractable open sets

Fix a positive integer n.

(C.2) If U is a nonempty contractable open subset of $\mathbf{R}^n$, is U necessarily homeomorphic to the open unit ball in $\mathbf{R}^n$?

For the record, to say that U is contractable means that the identity mapping on U is homotopic to a constant, through (continuous) mappings from U into itself. In particular, the homotopy and homology groups of U (of positive dimension) would then vanish, just as for an n-dimensional ball.

When $n = 1$, the answer to the question in (C.2) is "yes". In this case, U is either the whole real line, an open segment in the real line, or an open ray. Each of these is easily seen to be homeomorphic to the interval $(-1, 1)$, which is the unit ball in this case.

If $n = 2$, then the answer to the question in (C.2) is "yes" again. This is a well-known fact, and we shall return to it later, in Subsection C.8.

Starting in dimension 3, the answer to the question in (C.2) is "no". We shall say something about examples for this in a moment, but let us first ask ourselves the following: how might one be able to tell that a given contractable open set in $\mathbf{R}^n$ is *not* homeomorphic to an n-dimensional ball?

Here again a localized version of the fundamental group is important. If $n \geq 3$, then a necessary condition for a set U to be homeomorphic to an n-dimensional ball is that U be "simply connected at infinity". Roughly speaking, this means that if one takes a closed loop γ out near infinity in U, then it should be possible to contract γ to a point, while staying out near infinity too (although perhaps not as much as γ itself is).

Here is a more formal definition. For this we also include "connectedness at infinity" as a first part.

Definition C.3 *Let U be an open set in $\mathbf{R}^n$, or a topological space more generally. (Normally one might at least ask that U be locally compact.)*

U is connected at infinity *if for each compact set $K_0 \subseteq U$ there is a larger compact set $L_0 \subseteq U$ such that every pair of points in $U \backslash L_0$ is contained in a connected set which is itself contained in $U \backslash K_0$. (One can define "arcwise connectedness at infinity" in a similar manner. The two notions are equivalent under assumptions of local arcwise connectedness, and for topological manifolds in particular.)*

U is simply-connected at infinity *if it is connected at infinity, and if for every compact set $K_1 \subseteq U$ there is a larger compact set $L_1 \subseteq U$ so that if γ is an arbitrary closed loop in $U \backslash L_1$ (i.e., an arbitrary continuous mapping from the unit circle $\mathbf{S}^1$ into $U \backslash L_1$), then γ is homotopic to a constant through continuous mappings from the circle into $U \backslash K_1$.*

If U is the unit ball in $\mathbf{R}^n$, $n \geq 3$, then U is simply-connected at infinity. Indeed, let $B(0, r)$ denote the open ball in $\mathbf{R}^n$ with center 0 and radius r,

and let $\overline{B}(0,r)$ denote the corresponding closed ball. Then every compact subset of $B(0,1)$ is contained in $\overline{B}(0,r)$ for some $r < 1$. For each $r < 1$, $\overline{B}(0,r)$ is a compact subset of $B(0,1)$, and $B(0,1) \backslash \overline{B}(0,r)$ is connected when $n \geq 2$, and simply-connected when $n \geq 3$. This is because $B(0,1) \backslash \overline{B}(0,r)$ is homeomorphic to $\mathbf{S}^{n-1} \times (r,1)$, and $\mathbf{S}^j$ is connected when $j \geq 1$, and simply-connected when $j \geq 2$.

The property of being simply-connected at infinity is clearly preserved by homeomorphisms. Thus to get a contractable open set U in $\mathbf{R}^n$ which is *not* homeomorphic to an n-dimensional ball, it suffices to choose U so that it is not simply-connected at infinity.

If U is contractable, then it is simply-connected itself in particular. If U is simply-connected, connected at infinity, and not simply-connected at infinity, then it means that there is a compact set $K \subseteq U$ and loops γ in U which lie as far towards infinity as one would like (i.e., in the complement of any given compact subset of U) such that (a) γ can be contracted to a point in U, and (b) γ cannot be contracted to a point in $U \backslash K$. To put it another way, these loops γ can be contracted to points in U, but in doing this one always has to pass through at least one element of the compact set K.

A mechanism for having this happen for a set U contained in $\mathbf{R}^3$ is given by the construction of "the Whitehead continuum" [White]. (See also [Dave2, Kir].) Here is an outline of the procedure.

Start with a standard smooth "round" solid torus T in $\mathbf{R}^3$. Here T should be a compact set, i.e., it should contain its boundary.

Next one chooses another smooth solid torus T_1 inside T. More precisely, T_1 should lie in the interior of T. One chooses T_1 in a particular way, which can be imagined as follows. (Pictures can be found on p68 of [Dave2] and p82 of [Kir].) First take a "small" solid torus in T, small enough to be contained in a topological ball in T. One can think of grabbing hold of this small solid torus at two ends, and then stretching them around the "hole" in the larger torus T. One stretches them around the two different sides of the hole in T. To get T_1, these two ends should hook around each other on the other side of the hole. In other words, one might imagine having the two ends of the small solid torus from before, stretched around opposite sides of T, and then passing one across the other, until they do not touch any more, but are clasped together, like two hooks, or two links in a chain. The configuration looks *locally* like two hooks or links clasped together, but in fact one has two ends of the single solid torus T_1, wrapped around the hole in T.

If T_1 is chosen in this way, then it has the following two basic properties.

The first is that it is *homotopically* trivial in T. That is, the identity mapping on T_1 is homotopic to a constant mapping through (continuous) mappings from T_1 into T. This follows exactly the description above; in making the homotopy, one is allowed to stretch or move T_1 around as much as one like, and one is allowed to have *different parts of (images of)* T_1 *cross each other in* T. To put it a bit differently, the mappings being deformed are *not* required to be injective.

The second property is that T_1 is not "isotopically trivial". This means that one cannot continuously deform T_1 through an isotopy of T into a set which lies in a ball contained in T. In effect, this means that one cannot continuously deform T_1 inside T in such a way that T_1 ends up in a ball in T, and so that the deformations do *not* ever cross each other (unlike the homotopy in the previous paragraph). If one could get T_1 inside a ball in T, then one could continue the deformation to get an isotopy into an arbitrarily small ball. One would not ask for shrinking T_1 to a point here, because this is automatically prevented by injectivity (independent of clasping or not).

This explains how T and T_1 should be chosen. Since T_1 is a 3-dimensional smooth solid torus in its own right, one can repeat the process to get another smooth solid torus T_2 contained in it, and in fact contained in the interior of T_1. In other words, since T and T_1 are both smooth solid tori, they are diffeomorphic to each other in particular, and this can be used to make precise the idea of "repeating the process". Specifically, if $\phi : T \to T_1$ is such a diffeomorphism, then one can take T_2 to be $\phi(T_1)$.

One then repeats the process indefinitely, getting smooth solid tori T_j for $j = 1, 2, \ldots$ such that T_{j+1} is contained in the interior of T_j for each j, and so that T_{j+1} is arranged in T_j in the same way as T_1 is arranged in T.

Now let W be the intersection of all these solid tori T_j. This gives a nonempty compact set in $\mathbf{R}^3$. We can think of W as lying inside of $\mathbf{S}^3$, and then take $U = \mathbf{S}^3 \backslash W$. One can also rotate this around so that U actually lies in $\mathbf{R}^3$. One can show that U is contractable, but not simply-connected at infinity. See [Dave2, Kir, White] for more information. (For the purposes of looking at U, the complement of W, it can be convenient to use a modestly different description of the construction, in which one builds U up from smaller pieces in an "increasing" manner, analogous to the "decreasing" construction for W above.)

Although U is not homeomorphic to a 3-dimensional ball in this case, the Cartesian product of U with a nonempty open interval is homeomorphic to a 4-dimensional ball. This is attributed to Arnold Shapiro in [Bin3]; see also

Section 10 of [Bin4] and [Kir]. This is analogous to the effect of stabilization before, in Subsection C.1. In particular, one can check directly that taking the Cartesian product with the interval gets rid of the problem that U itself has with simple-connectivity at infinity. This is a general phenomenon, which is relevant as well for other situations mentioned in this appendix. A similar point came up Subsection C.1.

Beginning in dimension 4, there are contractable open sets in $\mathbf{R}^n$ which are not topological n-balls, and which have the additional feature that their closures are compact manifolds with boundary. This last does not work in dimension 3, and, for that matter, the complement of the Whitehead continuum in $\mathbf{S}^3$ cannot be realized as the interior of a compact manifold with boundary, whether or not this compact manifold should occur as the closure of the set in $\mathbf{S}^3$. The reason is that if such a compact manifold did exist, its boundary would be a 2-dimensional surface with the homology of the 2-sphere. We shall say more about this in a moment. In this case the boundary would have to be homeomorphic to the 2-sphere. This would contradict the failure of simple-connectivity at infinity for the original space, since $\mathbf{S}^2$ is simply-connected.

The difference with $n \geq 4$ is that the boundary can be a homology $(n-1)$-sphere (i.e., a manifold with the same homology as $\mathbf{S}^{n-1}$) which is not simply-connected. The interior then fails to be simply-connected at infinity again, and is not homeomorphic to an n-ball in particular.

For some related information and references concerning these examples in dimensions greater than or equal to 4, see [Dave2], including the top of p94, and the discussion on p103-104.

C.2.1 Some positive results

For dimensions $n \geq 4$, it is known that every contractable topological manifold M which is simply-connected at infinity is homeomorphic to $\mathbf{R}^n$. See [Sta1] for $n \geq 5$, and Corollary 1.2 on p366 of [Fre] for $n = 4$. A related reference is [McMZ]. Actually, [Sta1] is stated for the piecewise-linear category; one can go from there to the topological category via [KirS]. The four-dimensional result does not work in the smooth or piecewise-linear categories (which are equivalent in dimension 4), because of the existence of "fake $\mathbf{R}^4$'s (smooth manifolds homeomorphic to $\mathbf{R}^4$, but not diffeomorphic to it). Concerning the latter, see [FreQ] (p122 in particular) and [Kir] (Chapter XIV).

These topics are also related to "McMillan's cellularity criterion", in [McM]. A 4-dimensional version of this is given in [Fre], in Theorem 1.11 on p373. We shall discuss cellularity and this criterion further in Subsubsection C.4.1.

Now let us look more closely at the case of compact manifolds with boundary. Suppose that N is an n-dimensional compact topological manifold with boundary. Consider the following question:

(C.4) If N is contractable and ∂N is a topological $(n-1)$-sphere, is N homeomorphic to the closed unit ball in $\mathbf{R}^n$?

This question is actually equivalent to the Poincaré conjecture (in dimension n, and in the topological category). This is a well-known fact. The main points are the following. If one is given a compact n-dimensional topological manifold without boundary which is a homotopy n-sphere, then one can get an n-dimensional manifold N as in (C.4) from it by cutting out a topological ball (with tame boundary). If this manifold N is homeomorphic to the closed unit ball in $\mathbf{R}^n$, then one can obtain that the original space was homeomorphic to the standard n-sphere, by gluing the ball which was removed back in. (We shall say more about this in Remark C.5.) Conversely, given a manifold N as in (C.4), one can get a homotopy n-sphere from it by gluing in a ball along the boundary of N. To go from this and the Poincaré conjecture to the conclusion that N is homeomorphic to a closed ball, one can use the "generalized Schönflies theorem", discussed later in this subsection (after Remark C.5).

In particular, the answer to (C.4) is known to be "yes" when $n \neq 3$, and the problem is open for $n = 3$.

There are some analogous relationships between the Poincaré conjecture and contractable open manifolds. Namely, if one starts with a compact n-dimensional topological manifold without boundary P which is a homotopy n-sphere, then one can get an n-dimensional contractable open manifold by removing a point x from P. If $n \geq 3$, then $P \backslash \{x\}$ will also be simply-connected at infinity, as one can check using the manifold structure of P around x. If one knows that $P \backslash \{x\}$ is homeomorphic to $\mathbf{R}^n$, then one can deduce that P, which is topologically the same as the one-point compactification of $P \backslash \{x\}$, is homeomorphic to $\mathbf{S}^n$.

However, it is not as easy to go in the other direction, from contractable open manifolds which are simply-connected at infinity to compact manifolds

which are homotopy-equivalent to a sphere, as it is in (C.4). One can take the one-point compactification of the open manifold to get a compact space, but it is not immediately clear that this space is a manifold. The simple-connectivity at infinity for the open manifold is a necessary condition for this (when $n \geq 3$), but the converse is more complicated. There are broader issues concerning the behavior of open manifolds at infinity, and we shall mention some aspects of this in Subsubsection C.2.2 and Subsection C.3.

For the first part, about going from P to an open manifold, suppose that one is in a situation where there is a general result to the effect that an n-dimensional contractable open manifold which is simply-connected at infinity is homeomorphic to $\mathbf{R}^n$ for some fixed n. As above, one can use this to show that a compact n-dimensional manifold P (without boundary) which is homotopy-equivalent to $\mathbf{S}^n$ is homeomorphic to $\mathbf{S}^n$. A complication with this type of argument is that one does not necessarily say too much about the behavior of the homeomorphism at the point x which was removed and added back again (in the notation before), even if one knows more about P and the homeomorphism between $P \backslash \{x\}$ and $\mathbf{R}^n$. In this respect, arguments that go through compact manifolds with boundary, as in (C.4), can work better; there are also some tricky aspects in this case, though, and we shall say more about this next.

Remark C.5 There are some subtleties about gluing in balls in the context of (C.4) and its correspondence with the Poincaré conjecture in the *smooth* category. If one takes two copies of the closed unit ball in $\mathbf{R}^n$, and glues them together using a homeomorphism between their boundaries, then the resulting space is homeomorphic to a standard n-dimensional sphere. This is a standard observation (which can be proved using the fact mentioned in the next paragraph), and it works for any gluing homeomorphism. If the gluing map is a diffeomorphism, then the resulting space is a smooth manifold in a natural way, but it may not be diffeomorphic to a standard sphere. Exotic spheres can be viewed in this manner, as gluings of standard closed balls through (tricky) diffeomorphisms along their boundaries.

In the topological case, one can use the following fact. Let B_n denote the closed unit ball in $\mathbf{R}^n$. If h is a homeomorphism from ∂B_n onto itself, then h can be extended to a homeomorphism from B_n onto itself. One can do this by a straightforward "radial extension". This method also works for the analogous statement in the piecewise-linear category. However, in the smooth category, a radial extension like this is not smooth in general at the

origin in B_n. An extension to a diffeomorphism may simply not exist (radial or not).

In any of the three categories, once one has an extension like this, one can use it to get an equivalence between the space obtained by gluing together the two copies of B_n, and the standard n-dimensional sphere. The extension unwinds the effect of the gluing map, if the gluing map is not the standard one. In the smooth case, this may not be possible, and this occurs with exotic spheres.

Let us look some more at (C.4), in the topological category. If N happens to be given as a subset of $\mathbf{R}^n$, in addition to the conditions in (C.4), then N is homeomorphic to the closed unit ball in $\mathbf{R}^n$. This can be derived from the "generalized Schönflies theorem" [Brow1, Maz, Mors]. This result says that if one has an embedding f of $\mathbf{S}^{n-1} \times [-1, 1]$ into $\mathbf{R}^n$, then $f(\mathbf{S}^{n-1} \times \{0\})$ can be realized as the image of $\mathbf{S}^{n-1}$ under a homeomorphism mapping all of $\mathbf{R}^n$ onto itself. See also Theorem 6 on p38 of [Dave2].

Let us be a bit more precise about the way that the generalized Schönflies theorem is used here. The first point is that the boundary of N is "collared" in N. This means that there is a neighborhood of ∂N in N which is homeomorphic to $\partial N \times [0, 1)$, and where the homeomorphism maps each point $z \in N$ to $(z, 0) \in \partial N \times \{0\}$. The assumption that N be a topological manifold with boundary gives a local version of this at each point in ∂N, and one can derive the existence of a global collaring from a result of Brown. See [Brow2, Con] and Theorem 8 on p40 of [Dave2].

On the other hand, to apply the generalized Schönflies theorem, one needs a topological $(n-1)$-sphere in $\mathbf{R}^n$ which is "bicollared", i.e., occurs as $f(\mathbf{S}^{n-1} \times \{0\})$ for some embedding $f : \mathbf{S}^{n-1} \times [-1, 1] \to \mathbf{R}^n$. The boundary of N is collared inside of N, but may not be bi-collared inside $\mathbf{R}^n$. To deal with this, one can use a parallel copy of ∂N in the interior of N, provided by the collaring of ∂N inside N. This parallel copy is now bi-collared in the interior of N, because of the collaring that we have for N.

If N lies inside $\mathbf{R}^n$, then this parallel copy is also bi-collared inside $\mathbf{R}^n$. One can apply the generalized Schönflies theorem, to get that the region in $\mathbf{R}^n$ bounded by this parallel copy of ∂N, together with this copy of ∂N itself, is homeomorphic to the closed unit ball in $\mathbf{R}^n$. To get back to N in its entirety, one uses the original collaring of ∂N inside N, to know that the missing part of N is homeomorphic to the product of $\partial N \cong \mathbf{S}^{n-1}$ with an interval, and to glue this to the other piece without causing trouble.

Thus one can get a positive answer to (C.4) when N lies inside $\mathbf{R}^n$, using the generalized Schönflies theorem. This is simpler than the solutions of the Poincaré conjecture, and it does not require any restrictions on the dimension n. The assumption of contractability of N is not needed for this either. (For arbitrary manifolds, not necessarily embedded in $\mathbf{R}^n$, this assumption would be crucial.)

In general, if N is an n-dimensional compact topological manifold with boundary which is contractable, then the boundary ∂N is always a homology sphere (has the same homology groups as an $(n-1)$-dimensional sphere). This is a well-known fact. One could use Theorem 9.2 on p357 of [Bred1], for instance. Conversely, any compact $(n-1)$-dimensional topological manifold without boundary which is a homology sphere can be realized as the boundary of an n-dimensional compact topological manifold with boundary which is contractable. This is elementary for $n \leq 3$, where the homology spheres are all ordinary spheres, and can be filled with balls. For $n \geq 5$, this is given in [Ker1], in the piecewise-linear category (for both the homology sphere and its filling by a contractable manifold). For $n \geq 6$, one can convert this into a statement about topological manifolds, through the Kirby–Siebenmann theory [KirS]. See also the bottom of p184 of [Dave2]. For $n = 5$ in the topological category, see the corollary on p197 of [FreQ]. See also Corollary 2B on p287 of [Dave2] for $n \geq 5$ and the topological category. (For the smooth category in high dimensions, there are complications which come from the existence of exotic spheres, as in the discovery of Milnor.)

For $n = 4$, see [Fre, FreQ]. In particular, see Theorem 1.4' on p367 of [Fre], and Corollary 9.3C on p146 of [FreQ]. In this case it can happen that the filling by a contractable manifold cannot be given as a piecewise-linear manifold. The boundary ∂N would always admit a unique piecewise-linear structure, by well-known results about 3-dimensional manifolds (as in [Moi]). Concerning the possible lack of piecewise-linear filling for a homology 3-sphere by a contractable 4-manifold, see [Fre, FreQ, Kir].

A famous example of a homology 3-sphere which is not simply-connected is given by the "Poincaré homology sphere". This is a quotient of the standard $\mathbf{S}^3$ by the (finite) icosahedral group. See Theorem 8.10 on p353 of [Bred1]. This is a particular example where a contractable filling exists among topological 4-manifolds, but not among piecewise-linear manifolds. See [Fre, FreQ, Kir].

If H is a k-dimensional compact manifold (without boundary) which is k-dimensional homology sphere, and if H is also simply-connected, then H is

homotopy-equivalent to the standard k-dimensional sphere. This is a standard fact from topology, which was also mentioned in Section 3. In this case, the Poincaré conjecture in dimension k would seek to say that H should be homeomorphic to $\mathbf{S}^k$.

Note that if N is a compact manifold with boundary, then ∂N is simply-connected if and only if the interior of N is simply-connected at infinity in the sense of Definition C.3.

C.2.2 Ends of manifolds

Suppose that M is an n-dimensional manifold without boundary which is "open", i.e., not compact. What can one say about the "ends" of M?

In particular, when can M be realized as the interior of a compact manifold with boundary? This would be a nice way of "taming" the end.

This type of issue is clearly related to the questions considered throughout this subsection. It also makes sense in general, whether or not M is contractable, or one expects it to be homeomorphic to a ball, or one expects the end to be spherical.

Some sufficient conditions for realizing an open manifold as the interior of a compact manifold with boundary in high dimensions are given in [BroLL]. A characterization for this is given in [Sie1]. See also [Ker2] concerning the latter.

For dimension 5 (with 4-dimensional boundaries), see [Qui3] and Section 11.9 of [FreS]. For dimension 4 (with 3-dimensional boundaries), see Theorem 1.12 on p373 of [Fre], and Section 11.9 in [FreQ]. Concerning dimension 3, see p216 of [FreQ].

In all of these, the fundamental group at infinity plays an important role.

C.3 Interlude: looking at infinity, or looking near a point

Let M be a topological manifold of dimension n, and without boundary. Assume that M is *open*, i.e., not compact.

Define $\widehat{M}$ to be the one-point compactification of M, through the usual recipe. That is, one adds to M a special point q, the point at infinity, and the neighborhoods of q in $\widehat{M}$ are given by sets of the form $\widehat{M} \backslash K$, where K is a compact subset of M.

Consider the following question:

(C.6) Under what conditions is $\widehat{M}$ a topological manifold?

One might look at this as a kind of *local* question, about the behavior of a space at a given point, or as a question about *large-scale* behavior of M. It is not hard to see that $\widehat{M}$ will be a topological manifold exactly when M looks like (is homeomorphic to) $\mathbf{S}^{n-1} \times [0,1)$ outside of a set with compact closure. Equivalently, $\widehat{M}$ is a manifold exactly when M can be realized as the interior of a compact manifold with boundary, where the boundary is homeomorphic to $\mathbf{S}^{n-1}$. (This uses Brown's theorem about the existence of collars for boundaries of manifolds with boundary, as in [Brow2, Con] and Theorem 8 on p40 of [Dave2].)

The large-scale perspective of (C.6) is somewhat close in outlook to Subsection C.2, especially Subsubsection C.2.1, while the local view is perhaps more like the perspective in Subsection C.1. Concerning the latter, one might think of local taming properties of embedded sets in terms of existence of "normal bundles" for the embedded sets. Similarly, one can think of (C.6) as asking about the existence of a normal bundle for $\widehat{M}$ at the point q. In this regard, one might compare with the discussion in Section 9.3 in [FreQ], especially Theorem 9.3A and Corollary 9.3B.

For the record, let us note that a necessary condition for $\widehat{M}$ to be a manifold is that

(C.7) M is simply-connected at infinity,

at least if $n \geq 3$. In "local" language, we can reformulate this condition as follows: for every neighborhood U of q in $\widehat{M}$, there is a neighborhood V of q such that $V \subseteq U$,

(C.8) every pair of points $x, y \in V \backslash \{q\}$ lies in a connected set in $U \backslash \{q\}$,

and

(C.9) every loop γ in $V \backslash \{q\}$ can be contracted to a point in $U \backslash \{q\}$.

This is similar to the localized fundamental group conditions mentioned in Subsection C.1.

Let us now think of $\widehat{M}$ as being any topological space, and not necessarily the one-point compactification of an open manifold. For a given point $q \in \widehat{M}$, one can still ask whether $\widehat{M}$ is an n-dimensional manifold at q, i.e., if there is

a neighborhood of q in $\widehat{M}$ which is homeomorphic to an open ball in $\mathbf{R}^n$. The necessary condition in the preceding paragraph still applies (when $n \geq 3$), concerning local simple-connectivity of $\widehat{M} \backslash \{q\}$ near q (as in (C.8) and (C.9)).

For the rest of this subsection, let us assume that $n \geq 3$. Note that there are special results for detecting manifold behavior in a space of dimension 1 or 2. This is reviewed in the introduction of [Fer4].

As a special case, imagine now that $\widehat{M}$ is a finite polyhedron of dimension n. Let L denote the codimension-1 link of q in $\widehat{M}$, as discussed in Section 3. Thus L is an $(n-1)$-dimensional finite polyhedron, and $\widehat{M}$ looks locally at q like a cone over L, as in Section 3. (As in Section 3, L is determined up to piecewise-linear equivalence, but not as a polyhedron.)

In order for $\widehat{M}$ to be an n-dimensional topological manifold in a neighborhood of q, the link L should be fairly close to a standard $(n-1)$-sphere. In particular, it is not hard to see that L should be homotopy-equivalent to $\mathbf{S}^{n-1}$. This implies that L should be connected and simply-connected, under our assumption that n is at least 3.

In fact, in the case where $\widehat{M}$ is a finite polyhedron, the connectedness and simple-connectedness of the link L around q are equivalent to the local connectivity and simple-connectivity conditions for $\widehat{M} \backslash \{q\}$ near q indicated above, with (C.8) and (C.9). This is not hard to see, and it is also rather nice. To put it a bit differently, imagine that one starts with the class of finite polyhedra, and then tries to go to more general contexts of topological spaces. The local connectivity and simple-connectivity conditions for $\widehat{M} \backslash \{q\}$ at q as described above provide a way to capture the information in the connectedness and simple-connectedness of the codimension-1 link at q in the case where $\widehat{M}$ is a polyhedron, in a manner that makes sense for arbitrary topological spaces, without special structure as one has for finite polyhedra.

These local connectedness and simple-connectedness conditions for $\widehat{M} \backslash \{q\}$ at q should be compared with local connectedness and simple-connectedness conditions for $\widehat{M}$ itself. If $\widehat{M}$ is a polyhedron, then any point q in $\widehat{M}$ automatically has the feature that there are arbitrarily small neighborhoods U of q in $\widehat{M}$ which can be contracted to q while staying near q, and, in fact, while staying in U itself. This is because $\widehat{M}$ looks locally like a cone at q.

In the next subsection we shall look at another case of this kind of "local manifold" question.

C.4 Decomposition spaces, 1

Let n be a positive integer, and let K be a nonempty compact subset of $\mathbf{R}^n$. One could also consider general manifolds instead of $\mathbf{R}^n$ here, but we shall generally stick to Euclidean spaces for simplicity. The main ideas come up in this case anyway.

Imagine shrinking K to a single point, while leaving the rest of $\mathbf{R}^n$ alone, and looking at the topological space that results. This can be defined more formally as follows. Let us write $\mathbf{R}^n/K$ for the set which consists of the points in $\mathbf{R}^n$ which do not lie in K, together with a single point which corresponds to K itself. In other words, this is where we shrink K to a single point. This set can be given a topology in a standard way, so that a subset U of $\mathbf{R}^n/K$ is open if and only if its inverse image back in $\mathbf{R}^n$ is open. Here "inverse image" uses the automatic quotient mapping $\mathbf{R}^n$ to $\mathbf{R}^n/K$. (In concrete terms, the inverse image of U in $\mathbf{R}^n$ means the set of points in $\mathbf{R}^n$ which correspond to elements of U, where one includes all points in K if the element of $\mathbf{R}^n/K$ associated to K lies in U.)

This type of quotient $\mathbf{R}^n/K$ is a special case of a "decomposition space". We shall discuss the general situation further in Subsection C.6, but this special case already includes a lot of interesting examples and phenomena.

Now let us consider the following question:

(C.10) Given K as above, when is $\mathbf{R}^n/K$ a topological manifold?

This is really a special case of the situation in Subsection C.3. For this it is better to use $\mathbf{S}^n$ instead of $\mathbf{R}^n$, so that $\mathbf{S}^n/K$ — defined in the same manner as above — is equivalent to the one-point compactification of $\mathbf{S}^n \backslash K$.

Let us consider some basic examples. If K consists of only a single point, then $\mathbf{R}^n/K$ is automatically the same as $\mathbf{R}^n$ itself, and there is nothing to do. If K is a finite set with more than one element, then it is easy to see that $\mathbf{R}^n/K$ is not a manifold. If we let q denote the point in $\mathbf{R}^n/K$ which corresponds to K, then $(\mathbf{R}^n/K)\backslash\{q\}$ does not enjoy the local connectedness property that it should if $\mathbf{R}^n/K$ were a manifold at q, as in (C.8) in Subsection C.3. More precisely, this local connectedness property for the complement of $\{q\}$ would be necessary only when $n \geq 2$. When $n = 1$, one does not have to have this local connectedness condition, but then $(\mathbf{R}^n/K)\backslash\{q\}$ would have too many local components near q for $\mathbf{R}^n/K$ to be a manifold at q. (That is, there would be more than 2 such local components.)

Now suppose that K is a straight line segment in $\mathbf{R}^n$. In this event, $\mathbf{R}^n \backslash K$ is homeomorphic to $\mathbf{R}^n$ again. This is not hard to check. This would also work if K were a standard rectangular cell of higher dimension in $\mathbf{R}^n$.

More generally, this works if K is a *tame cell* in $\mathbf{R}^n$, meaning the image of a standard rectangular cell under a homeomorphism of $\mathbf{R}^n$ onto itself. This follows automatically from the case of standard rectangular cells.

However, if one merely assumes that K is homeomorphic to a standard rectangular cell, then it is not necessarily true that $\mathbf{R}^n / K$ is a manifold! This is another aspect of wild embeddings, from Subsection C.1. We shall say more about this as the subsection goes on. A concrete example is given by taking K to be a copy of the Fox–Artin wild arc in $\mathbf{R}^3$. (Compare with [Fer4].)

Note that we are not saying that $\mathbf{R}^n / K$ is *always* not a manifold when K is wildly embedded. The converse is true, that K must be wildly embedded when $\mathbf{R}^n / K$ is not a manifold (and K is a topological cell). This is just a rephrasal of the remark above, that $\mathbf{R}^n / K$ is a manifold when K is a tamely embedded cell.

Here is a slightly more foolish example, which one might view as a generalization of the earlier comments about the case where K is a finite set with more than a single point. Imagine now that K is a copy of the j-dimensional sphere $\mathbf{S}^j$, $1 \le j \le n-1$. For this let us use a standard, smooth, round sphere; it is not a matter of wildness that we want to consider.

In this case $\mathbf{R}^n / K$ is *never* a topological manifold. If $j = n-1$, then $\mathbf{R}^n / K$ is homeomorphic to the union of $\mathbf{R}^n$ and an n-sphere, with the two meeting at a single point. This point is the one that corresponds to K in $\mathbf{R}^n / K$. Let us denote this point by q again, as above. In this case $(\mathbf{R}^n / K) \backslash \{q\}$ does not have the right local-connectedness property at q in order for $\mathbf{R}^n / K$ to be a manifold, as in (C.8) in Subsection C.3.

If $j = n-2$, then one runs into trouble with local simple-connectivity of $(\mathbf{R}^n / K) \backslash \{q\}$ at q, as in (C.9) in Subsection C.3. For this one might think about the special case where $n = 3$, so that K is a standard circle in $\mathbf{R}^3$. It is easy to take small loops in $\mathbf{R}^3 \backslash K$, lying close to K, which are nonetheless linked with K. These loops then project down into $(\mathbf{R}^3 / K) \backslash \{q\}$, where they can be as close to the point q as one likes, but they are never contractable in $(\mathbf{R}^3 / K) \backslash \{q\}$ at all, let alone in small neighborhoods of q (as in (C.9) in Subsection C.3). This is the same as saying that these loops are not contractable inside of $\mathbf{R}^n \backslash K$, which is equivalent to $(\mathbf{R}^n / K) \backslash \{q\}$.

When $j < n-2$, then one has similar obstructions to $\mathbf{R}^n / K$ being a

manifold, but in terms of the failure of higher-dimensional forms of local connectedness of $(\mathbf{R}^n/K)\backslash\{q\}$ (using homology or homotopy). This is analogous to the cases already described, when $j = n - 1$ or $n - 2$. We shall say more about this soon, but for the moment let us go on to some other matters.

For this example, where K is taken to be a standard j-dimensional sphere, note that $\mathbf{R}^n/K$ itself is locally contractable at q. This is as opposed to connectedness properties of $(\mathbf{R}^n/K)\backslash\{q\}$, and it is analogous to what happens in the case of finite polyhedra. Specifically, for finite polyhedra one always has local contractability, but the behavior near a given point of *punctured* neighborhoods around that point are another matter. The latter is connected to the behavior of the codimension-1 link of the polyhedron around the given point, as in Subsection C.3.

In the present case, where we have $\mathbf{R}^n/K$ with K a standard round j-dimensional sphere, one can see the local contractability of $\mathbf{R}^n/K$ at the point q (corresponding to K) as follows. In $\mathbf{R}^n$, one can take a tubular neighborhood of K, which is homeomorphic to the Cartesian product of the j-sphere K and an $(n - j)$-dimensional ball. This neighborhood can be contracted onto K in a simple way, and this leads to the local contractability of $\mathbf{R}^n/K$ at q.

Now let us consider the case of the *Whitehead continuum*, from Subsection C.2. We should not really say *the* Whitehead continuum here, as there is some flexibility in the construction, which can lead to the resulting set W not being pinned down completely. This ambiguity will not really cause trouble for us here, and we can work with any compact set W in $\mathbf{R}^3$ which is obtained as in the procedure described in Subsection C.2.

The set W has the feature of being *cell-like*, as in the following definition.

Definition C.11 (Cell-like sets) *A compact set K in $\mathbf{R}^n$ is said to be* cell-like *if K can be contracted to a point inside of any neighborhood U of itself in $\mathbf{R}^n$.*

Compare with [Dave2], especially p120. That the Whitehead continuum W is cell-like is not hard to see from the construction of W, as the intersection of a decreasing sequence of solid tori with certain properties. Specifically, for this the key point is that the ℓth solid torus can be contracted to a point inside the previous one.

If K is a topological cell, then K is contractable to a point inside of itself, without using the extra bit of room provided by a small neighborhood of

itself. This is also independent of the way that K might be embedded into some $\mathbf{R}^n$, i.e., wildly or tamely. In this respect, W is like a topological cell (and hence the name "cell-like" for the property in Definition C.11).

For W, it is *not* true that $\mathbf{R}^3/W$ is a topological manifold. If we let q denote the point in $\mathbf{R}^3/W$ corresponding to W, then $(\mathbf{R}^3/W)\backslash\{q\}$ is *not* locally simply-connected at q (in the sense of the condition in Subsection C.3, around (C.9)). In concrete terms, this means that there are loops in $\mathbf{R}^3\backslash W$ (which is the same as $(\mathbf{R}^3/W)\backslash\{q\}$) which lie as close to W as one likes (in their entirety), but which cannot be contracted to a point in $\mathbf{R}^3\backslash W$ while remaining reasonably close to W.

These loops can be described concretely, as meridians in the solid tori whose intersection gives W. The loop from the solid torus T_j can be filled with a disk inside T_j, but not without crossing the smaller torus T_{j+1}, or any of its successors. This comes back to the way that each $T_{\ell+1}$ is "clasped" inside of T_ℓ. See [Dave2, Kir] for more information (including Proposition 9 on p76 of [Dave2]).

In any event, the failure of the local simple-connectivity of $(\mathbf{R}^3/W)\backslash\{q\}$ at q is equivalent to $\mathbf{S}^3\backslash W$ not being simply-connected at infinity, as in Subsection C.2. This also follows the discussion in Subsection C.3, and the comment just after (C.10).

This case is quite different from the one of embedding round spheres in $\mathbf{R}^n$, as discussed before. More precisely, let us compare the situation with W and the example before where K is a standard circle inside of $\mathbf{R}^3$. For the latter, there are loops in $\mathbf{R}^3\backslash K$ which lie as close to K as one wants, and which are not contractable to a point in $\mathbf{R}^3\backslash K$ at all, let alone in a neighborhood of K. For W, one has that $\mathbf{S}^3\backslash W$ is contractable (as mentioned in Subsection C.2), and this implies that $\mathbf{R}^3\backslash W$ is simply-connected. (This is a straightforward exercise.) Thus these loops near W can be contracted to a point in $\mathbf{R}^3\backslash W$, if one allows oneself to go away from W for the contraction.

Here is another aspect of this. Although one has these loops in $\mathbf{R}^3\backslash W$ which lie near W but cannot be contracted to a point in $\mathbf{R}^3\backslash W$ while staying near W, these loops can be made *homologically* trivial in $\mathbf{R}^3\backslash W$ while staying near W. That is, one can fill the loops with surfaces inside $\mathbf{R}^3\backslash W$ while staying close to W, if one allows the surfaces to have handles (rather than simply being a disk, as in the case of homotopic triviality). This is something that one can easily see from the pictures (as in [Dave2, Kir]). The basic idea is that one can fill the loops with disks, where the disks stay close to W, but also pass through W (and so are not in $\mathbf{R}^3\backslash W$). However, one can avoid the

intersection with W by cutting out a couple of small holes in the disk, and attaching a handle to them which goes along the boundary of the solid torus in the next generation of the construction. Then W will stay inside this next solid torus, throughout the rest of the construction, and this surface gives a way of filling the loop without intersecting W (or being forced to go far away from it).

This kind of filling by surfaces does *not* work in the case where we take K to be a standard circle in $\mathbf{R}^3$. In this situation, we have loops in $\mathbf{R}^3 \backslash K$ which lie close to K, and which are linked *homologically* with the circle K. In other words, the *linking number* of the loop with K is nonzero, and this linking number is a *homological* invariant which would vanish if the loop could be filled with a surface without intersecting K. (For more about "linking numbers", see [BotT, Bred1, Fla, Spa].)

In this respect, the case of Whitehead continua is much more tame than that of embedded circles and spheres of other dimensions, even if it is still singular. We shall encounter other versions of this, now and further in this appendix.

Here is another feature of W, which distinguishes it from ordinary circles in $\mathbf{R}^3$ (or spheres in $\mathbf{R}^n$ more generally). Let us think of W now as lying in $\mathbf{R}^4$ rather than $\mathbf{R}^3$, through the inclusion of $\mathbf{R}^3$ in $\mathbf{R}^4$ by taking the fourth coordinate to be 0.

For $\mathbf{R}^4$, we have that $\mathbf{R}^4/W$ *is* a topological manifold (homeomorphic to $\mathbf{R}^4$). The basic point behind this is the following. In the realization of W as the intersection of a decreasing sequence of solid tori in $\mathbf{R}^3$, the ℓth solid torus was always "clasped" in the previous one (as in Subsection C.2, and [Dave2, Kir]). In $\mathbf{R}^4$, the extra dimension provides a lot of extra room, in such a way that this "clasping" is not really present any more. If T' is a solid torus which is embedded and clasped inside of another solid torus T in $\mathbf{R}^3$, one can "unclasp" T' in $\mathbf{R}^4$ by lifting one end up, bringing it around the hole in T, and leaving the other end alone. This is a standard observation, and it is analogous to the way that knots in $\mathbf{R}^3$ become unknotted in $\mathbf{R}^4$.

In other words, this procedure gives a way to make a deformation of $\mathbf{R}^4$, in which the solid torus T' is mapped to a set of small diameter, while not moving points some distance away at all. By contrast, back in $\mathbf{R}^3$, it is not possible to make an isotopy which shrinks T' to a set of small diameter, while leaving the points in the complement of the larger solid torus fixed. This is exactly because of the way that T' is "clasped" in T, so that it cannot be "unclasped" by an isotopy in T. When one has the extra dimension in $\mathbf{R}^4$,

one can "undo" the clasping, by lifting one end up and moving it around, as indicated above.

Once one has this kind of "shrinking" in $\mathbf{R}^4$, one can use this to show that $\mathbf{R}^4/W$ is homeomorphic to $\mathbf{R}^4$. One can do this directly, using shrinking homeomorphisms like this, and combinations of them, to make a mapping from $\mathbf{R}^4$ to itself which shrinks W to a point while remaining injective (and continuous) everywhere else. One puts homeomorphisms like this on top of each other, and deeper and deeper in the construction of W, until W itself is shrunk all the way to a point. The various solid tori T_j in the construction, of which W is the intersection, are made smaller and smaller in this process. The trick is to do this without shrinking everything, so that the mapping that results remains a homeomorphism on the complement of W.

This idea of shrinking can be given a general form, and is discussed in detail in [Dave2]. See also [Edw2, Kir].

By contrast, let us consider the case of a circle K in $\mathbf{R}^3$. If one views K as a subset of $\mathbf{R}^4$ in the same way, then $\mathbf{R}^4/K$ is still *not* a topological manifold. This follows from our earlier discussion about circles and spheres of higher dimensions inside of $\mathbf{R}^n$ in general. One also does not get a manifold by replacing $\mathbf{R}^4$ with $\mathbf{R}^m$ for larger m's.

Notice, however, that there is a kind of "improvement" that occurs in adding dimensions in this way. If K is a circle in $\mathbf{R}^3$, and if q denotes the point in $\mathbf{R}^3/K$ which corresponds to K, then $(\mathbf{R}^3/K)\backslash\{q\}$ is not locally simply-connected at q. For that matter, $(\mathbf{R}^3/K)\backslash\{q\} \cong \mathbf{R}^3\backslash K$ is not simply-connected at all. When one considers K as a subset of $\mathbf{R}^4$, and asks analogous questions for $\mathbf{R}^4/K$ (or $\mathbf{R}^4\backslash K$), then there is no longer any trouble with *simple*-connectivity. The basic underlying problem continues, though, in the form of 2-*dimensional* connectivity. This is not hard to see.

Similarly, if one views K as a subset of $\mathbf{R}^n$ for larger n, then the trouble with connectivity in lower dimensions goes away, but $(n-2)$-dimensional connectivity still does not work.

With the Whitehead continuum we are more fortunate. The problem with local simple-connectivity goes away when we proceed from $\mathbf{R}^3$ to $\mathbf{R}^4$, but difficulties with higher-dimensional connectivity do not then arise in their place. One should not be too surprised about this, since the Whitehead continuum is cell-like, while circles or spheres of higher dimension are not at all cell-like. In other words, with circles or spheres (and their complements in $\mathbf{R}^n$), there is some clear and simple nontrivial topology around, while the Whitehead continuum is much closer to something like a standard cell, which

causes less trouble. (One can look at this more precisely, but we shall not pursue this here.)

C.4.1 Cellularity, and the cellularity criterion

Now let us look at some general notions and results, concerning the possibility that $\mathbf{R}^n/K$ be a topological manifold (and, in fact, homeomorphic to $\mathbf{R}^n$).

Definition C.12 (Cellularity) *A compact set K in $\mathbf{R}^n$ (or, more generally, an n-dimensional topological manifold) is said to be* cellular *if it can be realized as the intersection of a countable family of sets B_i, where each B_i is a topological n-cell (or, equivalently, homeomorphic to the closed unit ball in $\mathbf{R}^n$), and if each B_{i+1} is contained in the interior of the preceding B_i.*

Compare with [Dave2], especially p35, [Edw2], and p44 of [Rus1]. Alternatively, a compact set K is cellular if and only if any neighborhood of K contains an open set which contains K and is homeomorphic to the standard n-dimensional ball.

Theorem C.13 *Let K be a compact subset of $\mathbf{R}^n$. Then $\mathbf{R}^n/K$ is a topological manifold if and only if K is cellular in $\mathbf{R}^n$. In this case, $\mathbf{R}^n/K$ is homeomorphic to $\mathbf{R}^n$.*

See Exercise 7 on p41 of [Dave2] for the first assertion, and Proposition 2 on p36 of [Dave2] for the second one. (Concerning the latter, see Section 5 in [Dave2] too. Note that some of the notation in Exercise 7 on p41 in [Dave2] is explained in the statement of Proposition 2 on p36 of [Dave2].) See also [Edw2], especially the theorem on p114, and p44ff of [Rus1].

For the record, let us mention the following.

Proposition C.14 *Let K be a compact subset of $\mathbf{R}^n$. If K is cellular, then K is cell-like. Conversely, if n is equal to 1 or 2, then K is cellular if it is cell-like.*

The fact that cellularity implies cell-likeness follows easily from the definitions. When $n = 1$, the converse is very simple, since connectedness implies that a set is an interval, and hence cellular. In $\mathbf{R}^2$, the argument uses special features of plane topology. See Corollary 4C on p122 of [Dave2].

In higher dimensions, cell-like sets need not be cellular. Examples are given by Whitehead continua, and some wild embeddings of cells. However, there is an exact characterization of cellular sets among cell-like sets, which is the following. Basically, the point is to include the same kind of localized simple-connectivity of $\mathbf{R}^n \backslash K$ around K as discussed before.

Theorem C.15 *Let K be a compact set in $\mathbf{R}^n$, with $n \geq 3$. Then K is cellular inside of $\mathbf{R}^n$ if and only if* (a) *it is cell-like, and* (b) *for every open neighborhood U of K in $\mathbf{R}^n$ there is another open neighborhood V of K so that every continuous mapping from $\mathbf{S}^1$ into $V \backslash K$ can be contracted to a point inside of $U \backslash K$.*

This characterization of cellularity is stated in Theorem 5 on p145 of [Dave2]. This uses also the definition of the *cellularity criterion* given on p143 of [Dave2]. When $n \geq 4$, this result works for subsets of general n-dimensional topological manifolds, and not just $\mathbf{R}^n$. When $n = 3$, there is trouble with the general case of manifolds, related to the 3-dimensional Poincaré conjecture being unsettled; if the cellularity criterion holds for general manifolds, then the 3-dimensional Poincaré conjecture would follow, as discussed on p145 of [Dave2]. See Theorem 1.11 on p373 of [Fre] concerning the 4-dimensional case, and [McM] and Section 4.8 of [Rus1] for dimensions 5 and higher.

Corollary C.16 *Let K be a compact subset of $\mathbf{R}^n$, $n \geq 3$. If K is cell-like in $\mathbf{R}^n$, then $K \times \{0\}$ is cellular in $\mathbf{R}^{n+1}$.*

See Corollary 5A on p145 of [Dave2].

Corollary C.16 is analogous to the fact that wild embeddings into $\mathbf{R}^n$ can become tame when one passes from $\mathbf{R}^n$ as the ambient space to an $\mathbf{R}^m$ with $m > n$. This was discussed briefly in Subsection C.1, towards the end. Similarly, Theorem C.15 is analogous to taming theorems for embeddings mentioned in Subsection C.1.

Theorem C.15 and Corollary C.16 are also close to some of the matters described in Subsection C.2.

The main point behind the derivation of Corollary C.16 from Theorem C.15 is that by passing to a Euclidean space of one higher dimension, potential trouble with local simple-connectedness of the complement of K goes away. This fits with basic examples, and the Whitehead continuum in particular.

We saw before that when K is a sphere, the passage from $\mathbf{R}^n$ to $\mathbf{R}^{n+1}$ can get rid of the trouble with local simple-connectivity of the complement of K, but that problems remain with higher-dimensional connectivity of the complement. For cell-like sets, unlike general sets, it is only the localized 1-dimensional connectivity of the complement which is needed to get cellularity. This is shown by Theorem C.15.

In this regard, let us also notice the following simple converse to Corollary C.16.

Lemma C.17 *Suppose that K is a compact subset of $\mathbf{R}^n$. If $K \times \{0\}$ is cellular in $\mathbf{R}^{n+1}$, then K is cell-like in $\mathbf{R}^n$.*

Indeed, if $K \times \{0\}$ is cellular in $\mathbf{R}^{n+1}$, then it is also cell-like in $\mathbf{R}^{n+1}$, as in Proposition C.14. It is easy to check that cell-likeness for $K \times \{0\}$ in $\mathbf{R}^{n+1}$ implies cell-likeness for K inside $\mathbf{R}^n$, just by the definitions. (Thus cell-likeness, unlike cellularity, is not made more feasible by the extra room of extra dimensions.) This implies Lemma C.17.

For concrete examples of cell-like sets, often the cellularity in higher-dimensional spaces, as in Corollary C.16, can be seen in fairly direct and simple terms. The room from the extra dimensions makes it easy to move pieces of the set apart, without the claspings, knottings, etc., which occurred originally. Some aspects of this came up earlier, concerning Whitehead continua.

Before leaving this subsection, let us observe that the localized simple-connectivity conditions that are used here are a bit different from those employed in the context of taming theorems, as in Subsection C.1. To make this precise, let K be a compact subset of some $\mathbf{R}^n$. The conditions that come up in the present subsection involve the behavior of $\mathbf{R}^n \backslash K$, localized around K (the whole of K). That is, one looks at the behavior of $\mathbf{R}^n \backslash K$ within arbitrarily-small neighborhoods of K in $\mathbf{R}^n$. In the context of Subsection C.1, one would look at the behavior of $\mathbf{R}^n \backslash K$ near individual points in K.

To put it another way, here one seeks to contract loops in $\mathbf{R}^n \backslash K$ that are close to K to points, while staying close to K. In the context of Subsection C.1, one looks at *small* loops in $\mathbf{R}^n \backslash K$ near K, and tries to contract them to points in the complement while staying in *small balls*, and not just staying near K.

C.5 Manifold factors

Let W be a Whitehead continuum, constructed through a decreasing sequence of solid tori in $\mathbf{R}^3$, as in Subsection C.2.

Theorem C.18 *If $\mathbf{R}^3/W$ is defined as in Subsection C.4, then $(\mathbf{R}^3/W) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$.*

In particular, $(\mathbf{R}^3/W) \times \mathbf{R}$ is a topological manifold, even though $\mathbf{R}^3/W$ itself is not. Thus $(\mathbf{R}^3/W) \times \mathbf{R}$ is a *manifold factor*.

The fact that $(\mathbf{R}^3/W) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$ is given as Corollary 3B on p84 of [Dave2]. See also [AndR, Kir].

Note that the existence of a homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$ is *not* the same as the observation mentioned in Subsection C.4, that $\mathbf{R}^4/(W \times \{0\})$ is homeomorphic to $\mathbf{R}^4$. In considering $(\mathbf{R}^3/W) \times \mathbf{R}$, one is in effect taking $\mathbf{R}^4$, and then shrinking each copy $W \times \{u\}$ of W to a point, where u runs through all real numbers. For $\mathbf{R}^4/(W \times \{0\})$, one shrinks only a single copy of W to a point.

Although the construction is more complicated for $(\mathbf{R}^3/W) \times \mathbf{R}$ than for $\mathbf{R}^4/(W \times \{0\})$, there are some common aspects. As before, one of the main points is that the solid tori in $\mathbf{R}^3$ which are "clasped" (inside of other solid tori) become unclasped in $\mathbf{R}^4$. With the extra dimension in $\mathbf{R}^4$, one can pick up one end of one of these tori, bring it around, and then lay it down again, so that the clasping is undone. For the present situation with $(\mathbf{R}^3/W) \times \mathbf{R}$, one performs this kind of action for all of the copies $W \times \{u\}$ of W at once, $u \in \mathbf{R}$, rather than just a single copy. (Compare also with Subsection C.6, and the general notion of decomposition spaces mentioned there.)

In Subsection C.2, it was mentioned that $\mathbf{S}^3 \backslash W$ is a contractable open set which is *not* homeomorphic to a 3-ball (because it is not simply-connected at infinity), and that $(\mathbf{S}^3 \backslash W) \times \mathbf{R}$ *is* homeomorphic to a 4-dimensional open ball. (See [Bin3, Bin4, Kir].) This result is similar in some ways to Theorem C.18, but the conclusions are not quite the same either.

In this vein, let us make the following observation. As usual, denote by q the (singular) point in $\mathbf{R}^3/W$ that corresponds to W. Let us write L for the subset of $(\mathbf{R}^3/W) \times \mathbf{R}$ given by $q \times \mathbf{R}$. Thus L is homeomorphic to a line.

Using a homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ to $\mathbf{R}^4$, one gets an embedding of L into $\mathbf{R}^4$. It is not hard to see that any such embedding of L into $\mathbf{R}^4$ has to be wild. Just as $\mathbf{R}^3 \backslash W$ is not locally simply-connected near W, if $\mathcal{L}$ denotes the image of L in $\mathbf{R}^4$ by an embedding as above, then $\mathbf{R}^4 \backslash \mathcal{L}$ is

not locally simply-connected near $\mathcal{L}$. (Note that $\mathbf{R}^4 \backslash \mathcal{L}$ is homeomorphic to $(\mathbf{R}^3 \backslash W) \times \mathbf{R}$, by construction.) This ensures that $\mathcal{L}$ is wild in $\mathbf{R}^4$, no matter what homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$ one might use, since ordinary straight lines in $\mathbf{R}^4$ do not behave in this way.

One can also make local versions of this argument, to show that $\mathbf{L}$ is locally wild in the same manner.

If one were to want to pass from a homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$ in Theorem C.18 to a homeomorphism from $(\mathbf{S}^3 \backslash W) \times \mathbf{R}$ onto $\mathbf{R}^4$, then in particular one could be lead to try to figure out something about what happens when one deletes $\mathcal{L}$ from $\mathbf{R}^4$. Conversely, if one wanted to go in the other direction, one might have to figure out something about how to put the topological line back in. These endeavors should be at least somewhat complicated, because of the wildness of $\mathcal{L}$ inside of $\mathbf{R}^4$.

The next fact helps to give an idea of how wild $\mathcal{L}$ can have to be.

Theorem C.19 *Let U be an open set in some $\mathbf{R}^n$, and let F be a closed set in $\mathbf{R}^n$. If the Hausdorff dimension of F is less than $n-2$, then any open loop in $U \backslash F$ that can be contracted to a point in U can also be contracted to a point in $U \backslash F$. In particular, $\mathbf{R}^n \backslash F$ is simply-connected.*

In other words, if F is closed and has Hausdorff dimension $< n-2$, then F is practically invisible for considerations of fundamental groups, even local ones. From this one can check that $\mathcal{L}$ as above necessarily has Hausdorff dimension at least 2, no matter the homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$ which produced it. This is also true locally, i.e., each nontrivial arc of $\mathcal{L}$ has to have Hausdorff dimension at least 2, and for the same reasons. This is somewhat remarkable, since $\mathcal{L}$ is homeomorphic to a line (and straight lines have Hausdorff dimension 1). (In general, Hausdorff dimension need not be preserved by homeomorphisms, though, and this is an instance of that.)

Theorem C.19 is given (in a slightly different form) in [MartRiV3], in Lemma 3.3 on p9. See also [Geh2, LuuV, SieS, Väi3] for related results. In particular, [SieS] uses Theorem C.19 in a manner very similar to this, in the context of double-suspension spheres and homeomorphic parameterizations of them.

Note that instead of requiring that F have Hausdorff dimension less than $n-2$ in Theorem C.19, it is enough to ask that the $(n-2)$-dimensional Hausdorff measure of F be zero.

By now, there are many examples known of spaces which are manifold factors (and not manifolds). The original discovery was in [Bin3, Bin4], using

a different space. This example, Bing's "dogbone" space, will come up again in Subsection C.6.

As mentioned near the bottom of p93 of [Dave2], 3 is the smallest dimension in which this type of phenomenon can occur (where a non-manifold becomes a manifold after taking the Cartesian product with $\mathbf{R}$), because of results about recognizing manifolds in dimensions 1 and 2. It does occur in all dimensions greater than or equal to 3.

As a basic class of examples, if K is a compact set in $\mathbf{R}^n$ which is a cell, then $\mathbf{R}^n/K$ may not be a manifold if K is wild, but $(\mathbf{R}^n/K) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^{n+1}$. See [AndC, Bry1, Bry2, Dave2, Rus1].

More generally, if K is any compact set in $\mathbf{R}^n$ which is cell-like, then $(\mathbf{R}^n/K) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^{n+1}$. When $n = 1$ or 2, $\mathbf{R}^n/K$ is itself homeomorphic to $\mathbf{R}^n$. (Compare with Proposition C.14 and Theorem C.13 in Subsubsection C.4.1.) If $n \geq 3$, then $\mathbf{R}^n/K$ may not be homeomorphic to $\mathbf{R}^n$, but it is true that $(\mathbf{R}^n/K) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^{n+1}$ in this situation. See Proposition 2 on p206 of [Dave2] for $n = 3$, and Theorem 9 on p196 or Theorem 13 on p200 of [Dave2] for $n \geq 4$. (For Theorem 9 on p196 of [Dave2], note that the definition of a "k-dimensional decomposition" of a manifold is given near the top of p152 of [Dave2].)

Another class of examples (which is in fact closely related to the previous ones) comes from the celebrated double-suspension theorems of Edwards and Cannon [Can1, Can2, Can3, Dave2, Edw2] (mentioned in Section 3). From these one has the remarkable fact that there are *finite polyhedra* P which are not topological manifolds, but for which $P \times \mathbf{R}$ is a manifold.

There are compact sets K in $\mathbf{R}^n$ such that K is *not* cell-like, and not cellular in particular, but $(\mathbf{R}^n/K) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^{n+1}$. This happens for every $n \geq 4$. See Corollary 3E on p185 of [Dave2]. (The proof uses the double-suspension theorems.)

There are also non-manifold spaces which become manifolds after taking the product with other non-manifold spaces. See Section 29 of [Dave2], beginning on p223.

C.6 Decomposition spaces, 2

The construction of the quotient $\mathbf{R}^n/K$, given a compact subset K of $\mathbf{R}^n$, as in Subsection C.4, is an example of a *decomposition space*. More generally, one can allow many subsets of $\mathbf{R}^n$ (or some other space) to be contracted to points at the same time, rather than just a single set.

In general, a *decomposition* of $\mathbf{R}^n$, or some other space, means a partition of it, i.e., a collection of subsets which are pairwise disjoint, and whose union is the whole space. One can then form the corresponding quotient space, first as a set — by collapsing the sets in the partition to individual points — and then as a topological space. The topology on the quotient is the richest one (the one with the most open sets) so that the canonical mapping from the space to the quotient is continuous.

All of this makes sense in general, but in order to have some good properties (like the Hausdorff condition for the quotient space), some assumptions about the decomposition should be made. As a start, typical assumptions would be that the individual sets that make up the decomposition be closed, and that the decomposition satisfy a certain upper semi-continuity property. See [Dave2] for details, including the definition on p13 of [Dave2]. For the present discussion, we shall always assume that some conditions like these hold, even if we do not say so explicitly.

If G is a decomposition of $\mathbf{R}^n$, then one writes $\mathbf{R}^n / G$ for the corresponding quotient space.

Given a set K in $\mathbf{R}^n$, one can always consider the decomposition of $\mathbf{R}^n$ consisting of K and sets with only single elements, with the latter running through all points in $\mathbf{R}^n \backslash K$. This decomposition is sometimes denoted G_K, and the quotient $\mathbf{R}^n / G_K$ in the general sense of decompositions is the same as the space $\mathbf{R}^n / K$ from Subsection C.4.

As another basic situation, product spaces of the form $(\mathbf{R}^n / K) \times \mathbf{R}$ can be viewed as decomposition spaces. Specifically, one gets a decomposition of $\mathbf{R}^{n+1} \cong \mathbf{R}^n \times \mathbf{R}$ using sets of the form $K \times \{u\}$ in $\mathbf{R}^{n+1}$ for each $u \in \mathbf{R}$, together with single-element sets for all of the points in $\mathbf{R}^{n+1} \backslash (K \times \mathbf{R})$. The resulting decomposition space is equivalent topologically to $(\mathbf{R}^n / K) \times \mathbf{R}$.

An important general point is that wild or interesting embeddings can often occur in simple or useful ways through decompositions. For instance, in the decomposition space described in the preceding paragraph, one has a particular "line", corresponding to the copies of K. See [Can1, Dave1, Dave2] for more information, including p451 of [Can1], the last paragraph in Section 2 on p380 of [Dave1], and the remarks near the top of p37 in [Dave2].

The following theorem of R. L. Moore [Moo] is an early result about when decomposition spaces are manifolds, and homeomorphic to the original space.

Theorem C.20 *Let X be a compact Hausdorff topological space. Suppose that f is a continuous mapping from the 2-sphere $\mathbf{S}^2$ to X, and that for each*

$x \in X$, $f^{-1}(x)$ is nonempty and connected, and $\mathbf{S}^2 \backslash f^{-1}(x)$ is nonempty and connected. Then X is homeomorphic to $\mathbf{S}^2$.

In this theorem, the mapping f itself may not be a homeomorphism. As in Subsection C.4, f might have the effect of collapsing some line segments down to single points, for instance. It is true that f can always be approximated by homeomorphisms, however. See [Dave2] for more information and references.

Similar results hold in dimension 1. For this it is enough to assume that the inverse images of points under the mapping be *connected* (and nonempty proper subsets of the domain), without imposing conditions on their complements. In this situation, the inverse images of points will simply be arcs. Compare with [Dave2], including the remarks near the bottom of p17.

What might be reasonable analogues of Theorem C.20 in higher dimensions? One should not keep the hypotheses literally as they are above, where the fibers are connected and have connected complements, because of counterexamples like the non-manifold spaces that one can get by contracting a circle to a point (as in Subsection C.4).

However, in dimension 2, the property of a set in $\mathbf{S}^2$ or $\mathbf{R}^2$ being connected and having connected complement is quite strong. For a closed subset of $\mathbf{S}^2$ which is not empty nor all of $\mathbf{S}^2$, these conditions imply that the complement of the set is homeomorphic to a 2-dimensional disk, and that the set itself is *cellular* (Definition C.12 in Subsection C.4).

A decomposition G of $\mathbf{R}^n$ is said to be *cellular* if each of the subsets of $\mathbf{R}^n$ of which it is composed is cellular. (Compare with [Dave2], in the statement of Corollary 2A on p36.) As an analogy with Moore's theorem, one might hope that a quotient of $\mathbf{R}^n$ (or $\mathbf{S}^n$, or other topological manifolds) by a cellular decomposition is a manifold, and homeomorphic to $\mathbf{R}^n$ again (or to the original manifold, whatever it might be).

This is true for decompositions which consist of a single cellular subset of the space, together with all the remaining points in the space as one-element sets. In other words, this statement is true in the context of quotients as in Subsection C.4. See Theorem C.13 in Subsection C.4, and Proposition 2 on p36 of [Dave2].

For decompositions in general, it is not true that cellularity is sufficient to ensure that the quotient space is a manifold. This fails already for decompositions of $\mathbf{R}^3$. The first example of this was provided by Bing's "dogbone" construction in [Bin2]. See also [Dave2], especially Section 9, for this and other examples. For the statement that the decomposition space is not a

manifold, and not just not homeomorphic to $\mathbf{R}^3$, see Theorem 13 on 498 of [Bin2]. A recent paper related to this is [Arm].

The dogbone space was also used in the initial discovery of manifold factors. See [Bin3, Bin4].

Although quotients by cellular decompositions do not in general give manifolds, there are many nontrivial examples where this does occur, and results about when it should take place. A particularly nice and fundamental example is given by "Bing doubling". See [Bin1, Dave2] (Example 1 in Section 9 of [Dave2]). This is a decomposition of $\mathbf{R}^3$ for which the corresponding quotient space is homeomorphic to $\mathbf{R}^3$. While the quotient space is standard, the decomposition has some interesting features, giving rise to some wild embeddings in particular. This decomposition has a symmetry to it, which leads to a homeomorphic involution on $\mathbf{S}^3$ which is highly nonstandard. The fixed-point set of this homeomorphism is a wildly-embedded 2-sphere in $\mathbf{S}^3$. This construction apparently gave the first examples of involutions on $\mathbf{R}^3$ which were not topologically equivalent to "standard" ones, made from rotations, reflections, and translations. See [Bin1] for more information, especially Section 4 of [Bin1].

See also Section 9 of [Bin4] for a wild involution on $\mathbf{R}^4$, whose fixed point set is homeomorphic to the dogbone space. This uses the fact that the product of the dogbone space with the real line is homeomorphic to $\mathbf{R}^4$.

The results mentioned in Subsection C.5 — about quotients $\mathbf{R}^n/K$ being homeomorphic to a Euclidean space after taking the Cartesian product with $\mathbf{R}$ — can also be seen as providing nontrivial examples of cellular decompositions of $\mathbf{R}^{n+1}$ for which the corresponding quotients are manifolds, and are homeomorphic to $\mathbf{R}^{n+1}$.

C.7 Geometric structures for decomposition spaces

C.7.1 A basic class of constructions

One feature of decomposition spaces is that they do not a priori come with a canonical or especially nice *geometry*, or anything like that. The *topology* is canonical, but this is somewhat different. Note that there are general results about existence of metrics which are compatible with the topology, as in Proposition 2 on p13 of [Dave2]. Once one has one such metric, there are many which define the same topology. This is true just as well for ordinary Euclidean spaces, or the spheres $\mathbf{S}^n$, even if there are also special metrics

(like the Euclidean metric) that one might normally like or use.

In some cases (for decomposition spaces), there are some particularly nice or special geometries that one can consider. A number of basic examples — like the Whitehead continuum, Bing doubling [Bin1], and Bing's dogbone space [Bin2] — have a natural topological "self-similarity" to them, which can be converted into geometric self-similarity.

Let us be more precise. In these cases, the nondegenerate elements of the decomposition are generated by repeating a simple "rule". The "rule" can be described by a smooth domain D in $\mathbf{R}^n$ together with some copies of D embedded in the interior of D, in a pairwise-disjoint manner. To generate the nontrivial elements of the decomposition, one starts with D, and then passes to the copies of D inside of itself. For each of these copies of D inside of D, one can get a new collection of smaller copies of D, by applying the "rule" to the copies of D with which we started. One then repeats this indefinitely.

Thus, if the original "rule" involves m copies of D inside of itself, then the jth step of this process gives rise to m^{j-1} copies of D, with the first step corresponding to D alone.

The limiting sets which arise from this procedure are pairwise disjoint, and are used to define the decomposition. To do this carefully, one can think of the jth step of the process as producing a compact set C_j, which is the union of the m^{j-1} individual copies of D indicated above. The construction gives $C_j \subseteq C_{j-1}$ for all $j \geq 2$. To pass to the limit, one can take the set $C = \bigcap_{j=1}^{\infty} C_j$, and then use the components of C as subsets of $\mathbf{R}^n$ to be employed in the decomposition G of $\mathbf{R}^n$. For each point $x \in \mathbf{R}^n \backslash C$, one also includes the one-point set $\{x\}$ in the decomposition G.

The Whitehead continuum (discussed in Subsection C.2) is an example of this. There the "rule" is particularly simple, in that it is based on an embedding of a single solid torus T_1 inside a larger one T. At each stage of the process there is only one domain, and only one nondegenerate set being produced in the end (i.e., the Whitehead continuum). In particular, one can have $m = 1$ in the general set-up described above, and with the result being nontrivial. For Bing doubling and Bing's dogbone space, one has $m > 1$, i.e., there are more than one embedding being used at each step, and more than one copy of the basic domain. (For Bing doubling the basic domain is again a solid torus, while for the dogbone space it is a solid 2-handled torus. In Bing doubling one has $m = 2$, as the name suggests. For the dogbone space, $m = 4$.) In these cases the number of components grows exponentially in the process, and is infinite in the end (after taking the limit).

At any rate, there are a number of basic examples of decompositions generated in this manner, with individual copies of a domain being replaced systematically by some embedded subdomains, copies of itself, following repetitions of a single basic "rule". See [Dave2], especially Section 9.

In typical situations, the basic "rule" involves nontrivial distortion of the standard Euclidean geometry at each step. As a first point along these lines, when one embeds a copy of some (bounded) domain D into itself, then some change in the geometry (along the lines of shrinkage) is unavoidable, at least if the embedded copy is a proper subset of the original domain.

Given that there needs to be some shrinkage in the embedding, the next simplest possibility would be that the embeddings are made up out of something like dilations, translations, rotations, and reflections. In other words, except for a uniform scale factor, one might hope that the geometry does not have to change.

Normally this will not be the case. Some amount of bending or twisting, etc., will (in general) be involved, and needed, to accommodate the kind of topological behavior that is present. This includes linking, clasping, or things like that.

For the purpose of *choosing* a geometry that might fit with a given decomposition space, however, one can modify the usual Euclidean metric so that the embeddings involved in the basic "rule" do have the kind of behavior indicated above, i.e., a constant scale factor together with an isometry. The scale factors should be less that 1, to reflect the shrinking that is supposed to take place for the decomposition spaces (even at a purely topological level).

It is not hard to see that one can make deformations of geometry like this. One can do this in a kind of direct and "intrinsic" way, defining metrics on $\mathbf{R}^n$ with suitable properties. One can also do this more concretely, "physically", through embeddings of the decomposition spaces into higher-dimensional Euclidean spaces. In these higher-dimensional Euclidean spaces, the self-similarity that one wants, in typical situations, can be realized in terms of standard *linear* self-similarity, through dilations and translations.

More precisely, in these circumstances, the quotient of $\mathbf{R}^n$ by the decomposition can be realized topologically as an n-dimensional subset X of some $\mathbf{R}^N$ (with $N = n + 1$, for instance), in such a way that X is a smooth submanifold away from the natural singularities, and X is self-similar around these singularities.

To build such a set X, one can start with the complement of the original domain D in $\mathbf{R}^n$. One would view $\mathbf{R}^n \backslash D$ as an n-dimensional submanifold

of $\mathbf{R}^N$. In place of the iteration of the basic rule for the decomposition from before, one now stacks some "basic building blocks" in $\mathbf{R}^N$ on top of $\mathbf{R}^n \backslash D$ (along the boundary of D), and then on top of the other building blocks, over and over again.

These basic building blocks are given by n-dimensional smooth manifolds in $\mathbf{R}^N$ (with boundary). They are diffeomorphic to a single "model" in $\mathbf{R}^n$, which is the original domain D in $\mathbf{R}^n$, minus the interiors of the m copies of D embedded inside D, as given by the basic "rule" that generates the decomposition. The building blocks are all diffeomorphic to each other, since they are all diffeomorphic to this same model, but they are also constructed in such a way as to be "similar" to each other. That is, they can all be given by translations and dilations of each other. This is a key difference between this construction and the original decomposition in $\mathbf{R}^n$.

Further, the building blocks are constructed in such a way that their ends are all similar to each other (i.e., even different ends on the same building block). Specifically, the building blocks are chosen so that when one goes to stack them on top of each other, their "ends" fit together properly, with smoothness across the interfaces.

These things are not difficult to arrange. Roughly speaking, one uses the extra dimensions in $\mathbf{R}^N$ to straighten the "ends" in this way, so that the different building blocks can be stacked properly. Typically, this would involve something like the following. One starts with the basic model in $\mathbf{R}^n$, given by D minus the interiors of the m embedded copies of D in D. One then makes some translations of the m embedded subdomains in D, up into the extra dimension or dimensions in $\mathbf{R}^N$. Up there, these subdomains can be moved or bent around, until they are similar to D itself (i.e., being the same modulo translations and dilations). This can be done one at a time, and without changing anything near the boundary of the original domain D. In this manner, the original model region in $\mathbf{R}^n$ becomes realized as an n-dimensional compact smooth submanifold (with boundary) in $\mathbf{R}^N$, with the ends matching up properly under similarities.

To put it another way, the main "trade-off" here is that one gives up the "flatness" of the original model, as a region in $\mathbf{R}^n$, to get basic building blocks in $\mathbf{R}^N$ that are n-dimensional curved submanifolds whose ends are similar to each other. The curving of the interiors of these building blocks compensates for the straightening of their ends.

As above, one then stacks these building blocks on top of each other, one after another, to get a realization of the decomposition space by an n-

dimensional subset X of $\mathbf{R}^N$. (One also puts in some limiting points, at the ends of the towers of the building blocks that arise. In other words, this makes X be a *closed* subset of $\mathbf{R}^N$. These extra points are the singularities of X.) This subset is smooth away from the singularities, and self-similar at the singularities, because of the corresponding properties of the basic building blocks.

By choosing the scale-factors associated to the ends of the basic building blocks to be less than 1, the diameters of the ends tend to 0 (and in a good way) as one stacks the building blocks on top of each other many times. This corresponds to the fact that the sets in the decomposition are supposed to be shrunk to single points in the quotient space. This is also part of the story of the "limiting points" in the previous paragraph. The limiting points are exactly the ones associated (in the end) to the nondegenerate sets in the original decomposition in $\mathbf{R}^n$, which are being shrunk to single points.

The actual homeomorphic equivalence between the set X in $\mathbf{R}^N$ produced through this method and the decomposition space $\mathbf{R}^n/G$ with which one starts is obtained using the diffeomorphic equivalence between the building blocks in $\mathbf{R}^N$ and the original model in $\mathbf{R}^n$ (D minus the interiors of the m embedded copies of itself, as above). In rough terms, at the level of the topology, the same kind of construction is occurring in both places, X and the decomposition space, and one can match them up, by matching up the individual building blocks. This is not hard to track.

At any rate, more details for these various matters are given in [Sem7].

Instead of stacking basic building blocks on top of each other infinitely many times, one can stop after finitely many steps of the construction. This gives a set which is still smooth, and diffeomorphic to $\mathbf{R}^n$, but which approximates the non-smooth version that represents the decomposition space. To put it another way, although this kind of approximation is standard *topologically*, its *geometry* does reflect the basic rule from which the decomposition space is obtained. In particular, properties of the decomposition space can be reflected in questions of quantitative bounds for the approximations which may or may not hold.

Some versions of this come up in [Sem7]. In addition, there are some slightly different but related constructions, with copies of finite approximations repeated but getting small, at the same time that they include more and more stages in the stacking.

The topological structure of the decomposition space can also be seen in terms of Gromov–Hausdorff limits of smooth approximations like these.

When one makes constructions like these — either finite approximations or infinite limits — the self-similarity helps to ensure that the spaces behave geometrically about as well as they could. For instance, this is manifested in terms of properties like Ahlfors-regularity (Definition 5.1), and the local linear contractability condition (Definition 4.10), at least under suitable assumptions or choices. One also gets good behavior in terms of "calculus", like Sobolev and Poincaré inequalities. One has uniform rectifiability as well (Definition 5.4), although the spaces are actually quite a bit more regular than that. For smooth approximations with only finitely many levels in the construction, one can have uniform bounds, independent of the number of levels, for conditions like these. See [Sem7] for more information, and some slightly different versions of these basic themes.

To summarize a bit, although the quotient spaces that one gets from decompositions may not be topological manifolds, in many cases one can realize them geometrically in such a way that the behavior that occurs is really pretty good. With the extra structure from the geometry, there are extra dimensions to the story as a whole, and which are perhaps not apparent at first. (Possibilities for doing analysis on spaces like these, and spaces which are significantly different from standard Euclidean spaces at that, give one form of this.)

We shall look at these types of geometries from some more perspectives in the next subsubsections, and also consider some special cases.

These general matters are related as well to the topics of Appendix D.

C.7.2 Comparisons between geometric and topological properties

Many of the geometric properties that occur when one constructs geometries for decomposition spaces as above correspond in natural ways to common conditions in purely topological terms that one might consider.

For instance, if one starts with a decomposition space $\mathbf{R}^n/G$ from $\mathbf{R}^n$, and gives it a metric in which it becomes Ahlfors-regular of dimension n, then in particular the decomposition space has Hausdorff dimension n with respect to this metric. In the type of examples mentioned above, $\mathbf{R}^n/G$ would also have topological dimension n.

The Hausdorff dimension of a metric space is always greater than or equal to the topological dimension, as in Chapter VII of [HurW]. To have the two be equal is rather nice, and Ahlfors-regularity in addition even nicer. For the constructions described in Subsubsection C.7.1, Ahlfors-regularity

reflects the fact that one chooses the geometries to be smooth away from the singularities, and self-similar around the singularities.

For cell-like decompositions in general (of compact metric spaces, say), there is a theorem which says that the topological dimension of the quotient is less than or equal to the topological dimension of the space with which one starts, if the topological dimension of the quotient is known to be finite. See Theorem 7 on p135 of [Dave2], and see [Dave2] in general for more information. Note that it is possible for the topological dimension of the quotient to be infinite, by a result of Dranishnikov [Dra] (not available at the time of the writing of [Dave2]).

For the situations discussed in Subsubsection C.7.1, Ahlfors-regularity of dimension n for the geometry of the decomposition space is a nice way to have good behavior related to the dimension (and with the right value of the dimension, i.e., the topological dimension). In other words, it is a geometric property that fits nicely with general topological considerations (as in the preceding two paragraphs), while also being about as strong as it can be. It is also quantitative, and it makes sense to consider uniform bounds for smooth approximations that use only finitely many levels of the construction. (Note that it is automatically preserved under Gromov–Hausdorff limits, when there are uniform bounds. See [DaviS8] for more information.)

Another general result about decomposition spaces implies that a cell-like quotient of a space is *locally contractable* if the quotient has finite dimension, under suitable conditions on the space from which one starts (e.g., being a manifold). See Corollary 12B on p129 of [Dave2] (as well as Corollaries 4A and 8B on p115 and p119 of [Dave2] for auxiliary statements). Here *local contractability* of a topological space X means that for every point $p \in X$, and every neighborhood U of p in X, there is a smaller neighborhood $V \subseteq U$ of p in X such that V can be contracted to a point in U. If the quotient space is not known to be finite-dimensional, then there are weaker but analogous conclusions that one can make, along the lines of local contractability. See Corollary 11B on p129 of [Dave2].

These notions are purely topological ones. The local linear contractability condition (Definition 4.10) is a more quantitative version of this for metric spaces. It is about as strong as one can get, in terms of the sizes of the neighborhoods concerned. That is, if one tries to contract a small ball in the space to a point inside of another ball which is also small, but somewhat larger than the first one, then weaker conditions besides local linear contractability might allow somewhat larger radii for the second ball (than a

constant multiple of the initial radius). The *linear* bound fits naturally with self-similarity.

As indicated earlier (towards the end of Subsubsection C.7.1), local linear contractability does come up in a natural manner for the constructions in Subsubsection C.7.1. This fits nicely with the general topological results, i.e., by giving a geometric form which reflects the topological properties in about the best possible way.

Exactly how nice the spaces discussed in Subsubsection C.7.1 are depends on the features of the decompositions. This includes not only how many singularities there are, but also how strong they are. For instance, cellular decompositions can lead to nicer or more properties for the resulting spaces than cell-like decompositions. I.e., at the geometric level, cellularity can give rise to more good behavior than local linear contractability. For example, one can look at simple-connectivity of *punctured* neighborhoods in this regard, and things like that. This is something that one does not have for the space obtained by taking $\mathbf{R}^3$ and contracting a Whitehead continuum to a single point. Some other versions of this are considered in [Sem7].

C.7.3 Quotient spaces can be topologically standard, but geometrically tricky

We have seen before how decompositions of $\mathbf{R}^n$ might lead to $\mathbf{R}^n$ again topologically in the quotient, but do so in a manner that is still somehow nontrivial. For instance, the decomposition might arise from a nontrivial manifold factor, or lead to wild embeddings in the quotient which seem very simple (like a straight line) at the level of the decomposition. In these situations, one can still have highly nontrivial geometries from the procedures described in Subsubsection C.7.1, even though the underlying space is topologically equivalent to $\mathbf{R}^n$.

As a special case, the wild embeddings that can occur in the quotient (with the topological identification with a Euclidean space) can behave in a very special way in the geometric realization of the decomposition space that one has here, with geometric properties which are not possible in $\mathbf{R}^n$ with the standard Euclidean metric. We shall give a concrete example of this in a moment.

This is one way that geometric structures for decomposition spaces (as in Subsubsection C.7.1) can help to add more to the general story. One had before the notion that a wild embedding might have a simple realization

through a decomposition space, and now this might be made precise through metric properties of the embedding (like Hausdorff dimension), using the type of geometry for the decomposition space that one has here.

Here is a concrete instance of this. Let W be a Whitehead continuum in $\mathbf{R}^3$, as in Subsection C.2. Consider the corresponding quotient space $\mathbf{R}^3/W$, as in Subsection C.4. Through the type of construction described in Subsubsection C.7.1, one can give $\mathbf{R}^3/W$ a geometric structure with several nice properties. In short, one can realize it topologically as a subset of $\mathbf{R}^4$, where this subset is smooth away from the singular point, and has a simple self-similarity at the singular point. One could then use the geometry induced from the usual one on the ambient $\mathbf{R}^4$.

Let us write q for the singular point in $\mathbf{R}^3/W$, i.e., the point in the quotient which corresponds to W. Using the metric on $\mathbf{R}^3/W$ indicated above, let us think of $(\mathbf{R}^3/W) \times \mathbf{R}$ as being equipped with a metric. One could also think of $(\mathbf{R}^3/W) \times \mathbf{R}$ as being realized as a subset of $\mathbf{R}^5$, namely, as the product of the one in $\mathbf{R}^4$ mentioned before with $\mathbf{R}$.

In this geometry, $L = \{q\} \times \mathbf{R}$ is a perfectly nice line. It is a "straight" line! In particular, it has Hausdorff dimension 1, and locally finite length. However, the image of L inside of $\mathbf{R}^4$ under a homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$ will be wild. As in Subsection C.5, any such embedding of L in $\mathbf{R}^4$ must be wild, and in fact any such embedding must have Hausdorff dimension at least 2, with respect to the usual Euclidean metric in $\mathbf{R}^4$. This used Theorem C.19.

This shows that the geometry that we have for $(\mathbf{R}^3/W) \times \mathbf{R}$ has to be substantially different from the usual Euclidean geometry on $\mathbf{R}^4$, even though the two spaces are topologically equivalent. Specifically, even though there are homeomorphisms from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$, no such homeomorphism can be Lipschitz (with respect to the geometries that we have on these spaces), or even Hölder continuous of order larger than $1/2$.

Although $(\mathbf{R}^3/W) \times \mathbf{R}$ — with the kind of geometry described above — is quite different from $\mathbf{R}^4$ with the usual Euclidean metric, there is a strong and nice feature that it has, in common with $\mathbf{R}^4$. We shall call this property "uniform local coordinates".

Since $(\mathbf{R}^3/W) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$, it has homeomorphic local coordinates from $\mathbf{R}^4$ at every point. "Uniform local coordinates" asks for a stronger version of this, and is more quantitative. Specifically, around each metric ball B in $(\mathbf{R}^3/W) \times \mathbf{R}$ (with respect to the kind of geometry that we have), there are homeomorphic local coordinates from a standard Euclidean

ball β of the same radius in $\mathbf{R}^4$, such that

(C.21) the image of β under the coordinate mapping covers the given ball B in $(\mathbf{R}^3/W) \times \mathbf{R}$,

and

(C.22) the *modulus of continuity* of the coordinate mapping and its inverse can be controlled, uniformly over all choices of metric balls B in $(\mathbf{R}^3/W) \times \mathbf{R}$, and in a scale-invariant manner.

Here "modulus of continuity" means a function $\omega(r)$ so that when two points in the domain (of a given mapping) are at distance $\leq r$, their images are at distance $\leq \omega(r)$. Also, r would range through positive numbers, and $\omega(r)$ would be nonnegative and satisfy

$$\lim_{r \to 0} \omega(r) = 0. \tag{C.23}$$

This last captures the continuity involved, and, in fact, gives uniform continuity.

For a mapping from a compact metric space to another metric space, continuity automatically implies uniform continuity, which then implies the existence of some modulus of continuity. This is not to say that one knows much about the modulus of continuity, a priori. (One can always choose it to be monotone, for instance, but one cannot in general say how fast it tends to 0 as $r \to 0$.)

Concrete examples of moduli of continuity would include $\omega(r) = C\,r$ for some constant C, which corresponds to a mapping being Lipschitz with constant C, or $\omega(r) = C\,r^\alpha$, $\alpha > 0$, which corresponds to Hölder continuity of order α. One can have much slower rates of vanishing, such as $\omega(r) = (\log\log\log(1/r))^{-1}$.

In our case, with the property of "uniform local coordinates", we want to have a single modulus of continuity $\omega(r)$ which works simultaneously for all of the local coordinate mappings (and their inverses). Actually, we do not look at moduli of continuity for the mappings themselves, but renormalized versions of them. The renormalizations are given by dividing distances in the domain and range by the (common) radius of B and β. In this way, B and

β are viewed as though they have radius 1, independently of what the radius was originally. This gives a kind of uniform basis for making comparisons between the behavior of the individual local coordinate mappings.

Let us return now to the special case of $(\mathbf{R}^3/W) \times \mathbf{R}$, with the geometry as before. In this case one can get the condition of uniform local coordinates from the existence of topological coordinates (without uniform bounds), together with the self-similarity and smoothness properties of the set. Here is an outline of the argument. (A detailed version of this, for a modestly different situation, is given in [Sem7]. Specifically, see Theorem 6.3 on p241 in [Sem7]. Note that the property of uniform local coordinates is called "Condition $(**)$" in [Sem7], as in Definition 1.7 on p192 of [Sem7].)

Let B be a metric ball in $(\mathbf{R}^3/W)\times\mathbf{R}$, for which one wants to find suitable coordinates. Assume first that B does not get too close to the singular line $\{q\} \times \mathbf{R}$ in $(\mathbf{R}^3/W) \times \mathbf{R}$, and in fact that the radius of B is reasonably small compared to the distance from B to the singular line. In this case $(\mathbf{R}^3/W) \times \mathbf{R}$ is pretty smooth and flat in B, by construction (through the method of Subsubsection C.7.1). This permits one to get local coordinates around B quite easily, and with suitable uniform bounds for the moduli of continuity of the coordinate mappings and their inverses. The bounds that one gets are scale-invariant, because of the self-similarity in the geometric construction (from Subsubsection C.7.1). In fact, one can have Lipschitz bounds in this case, as well as stronger forms of smoothness.

If the ball B is reasonably close to the singular line $\{q\} \times \mathbf{R}$, then one can reduce to the case where it is actually centered on $\{q\} \times \mathbf{R}$. That is, one could replace B with a ball which is centered on $\{q\} \times \mathbf{R}$, and which is not too much larger. (The radius of the new ball would be bounded by a constant times the radius of B.) This substitution does not cause trouble for the kind of bounds that are being sought here.

Thus we suppose that B is centered on the line $\{q\} \times \mathbf{R}$. We may as well assume that the center of B is the point $(q, 0)$. This is because $(\mathbf{R}^3/W) \times \mathbf{R}$ and the geometry that we have on it are invariant under translations in the $\mathbf{R}$ direction, so that one can move the center to $(q, 0)$ without difficulty, if necessary.

Using the self-similarity of $(\mathbf{R}^3/W)\times\mathbf{R}$, one can reduce further to the case where the radius of B is approximately 1. For that matter, one can reduce to the case where it is equal to 1, by simply increasing the radius by a bounded factor (which again does not cause problems for the uniform bounds that are being considered here). (To be honest, if one takes the geometry for $\mathbf{R}^3/W$ to

be flat outside a compact set, as in Subsubsection C.7.1, then this reduction is not fully covered by self-similarity. I.e., one should handle large scales a bit differently. This can be done, and a similar point is discussed in [Sem7] in a slightly different situation (for examples based on "Bing doubling").)

Once one makes these reductions, one gets down to the case of the single ball B in $(\mathbf{R}^3/W) \times \mathbf{R}$, centered at $(q, 0)$ and with radius 1. For this single choice of scale and location, one can use the fact that $(\mathbf{R}^3/W) \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$ to get suitable local coordinates.

For this single ball B, there is no issue of "uniformity" in the moduli of continuity for the coordinate mappings. One simply needs *a* modulus of continuity.

A key point, however, is that when works backwards in the reductions just made, to go to arbitrary balls in $(\mathbf{R}^3/W) \times \mathbf{R}$ which are relatively close to the singular line $\{q\} \times \mathbf{R}$, one does get local coordinates with uniform control on the (normalized) moduli of continuity of the coordinate mappings and their inverses. This is because of the way that the reductions cooperate with the scaling and the geometry.

At any rate, this completes the outline of the argument for showing that $(\mathbf{R}^3/W) \times \mathbf{R}$ has uniform local coordinates, in the sense described before, around (C.21) and (C.22), and with the kind of geometry for $(\mathbf{R}^3/W) \times \mathbf{R}$ as before. As mentioned at the beginning, this argument is nearly the same as one given in [Sem7], in the proof of Theorem 6.3 on p241 of [Sem7].

It is easy to see that a metric space which is bilipschitz equivalent to some $\mathbf{R}^n$ has uniform local coordinates (relative to $\mathbf{R}^n$ rather than $\mathbf{R}^4$, as above). The required local coordinates can simply be obtained from restrictions of the global bilipschitz parameterization. The converse is not true in general, i.e., there are spaces which have uniform local coordinates and not be bilipschitz equivalent to the corresponding $\mathbf{R}^n$.

Examples of this are given by $(\mathbf{R}^3/W) \times \mathbf{R}$ with the kind of geometry as above. We saw before that the singular line $\{q\} \times \mathbf{R}$, which has Hausdorff dimension 1 in our geometry on $(\mathbf{R}^3/W) \times \mathbf{R}$, is always sent to sets of Hausdorff dimension at least 2 under any homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$, so that such a homeomorphism can never be Lipschitz, or even Hölder continuous of order greater than $1/2$.

However, one can also get much simpler examples, by taking snowflake spaces. That is, one can take $\mathbf{R}^n$ equipped with the metric $|x - y|^\alpha$, where $|x-y|$ is the usual metric, and α is a positive real number strictly less than 1. It is not hard to check that this has the property of uniform local coordinates.

It is not bilipschitz equivalent to $\mathbf{R}^n$, because it has Hausdorff dimension n/α instead of Hausdorff dimension n.

If one assumes uniform local coordinates relative to $\mathbf{R}^n$ and Ahlfors-regularity of dimension n (Definition 5.1), then the matter becomes much more delicate. Note that Ahlfors-regularity of the correct dimension rules out snowflake spaces, as in the previous paragraph. It is still not sufficient, though, because the $(\mathbf{R}^3/W) \times \mathbf{R}$ spaces with the type of geometry as above are Ahlfors-regular of dimension 4, in addition to having uniform local coordinates (relative to $\mathbf{R}^4$).

With Ahlfors-regularity of dimension n (and uniform local coordinates relative to $\mathbf{R}^n$), there are some relatively strong conclusions that one can make, even if one does not get something like bilipschitz parameterizations. Some versions of this came up in Sections 4 and 5, and with much less than uniform local coordinates being needed. We shall encounter another result which is special to dimension 2, later in this subsubsection.

Dimension 1 is more special in this regard (and as is quite standard). One can get bilipschitz equivalence with $\mathbf{R}$ from Ahlfors-regularity of dimension 1 and uniform local coordinates with respect to $\mathbf{R}$ (and less than that). This is not hard to do, using arclength parameterizations.

Let us note that in place of $(\mathbf{R}^3/W) \times \mathbf{R}$ in dimension 4, as above, one can take Cartesian products with more copies of $\mathbf{R}$ to get examples in all dimension greater than or equal to 4, with similar properties as in dimension 4. In particular, one would get sets which are Ahlfors-regular of the correct dimension, and which have uniform local coordinates, but not bilipschitz coordinates, or even somewhat less than bilipschitz coordinates.

In dimension 3, there is a different family of examples that one can make, based on "Bing doubling". This was mentioned earlier, and is discussed in [Sem7].

Notice that the uniform local coordinates property *would* imply a bilipschitz condition if the local coordinates all came from restrictions of a single global parameterization. In general, the uniform local coordinates property allows the local coordinate mappings to change as one changes locations and scales, and this is why it allows for the possibility that there is no global bilipschitz parameterization. The case of $(\mathbf{R}^3/W) \times \mathbf{R}$ provides a good example of this. For this one can check the earlier argument, and see how the local coordinates that are used are not restrictions of a single global parameterization. This is true even though the local coordinate mappings at different locations and scales often have a common "model" or behavior. In

other words, one might often use *rescalings* of a single mapping, and this can be quite different from *restrictions* of a single mapping.

Let us emphasize that in the uniform local coordinates condition, one does not ask that the uniform modulus of continuity be something simple, like a Lipschitz or a Hölder condition. Under some conditions one might be able to *derive* that, using the uniformity of the hypothesis over all locations and scales. This is analogous to something that happens a lot in harmonic analysis; i.e., relatively weak conditions that hold uniformly over all locations and scales often imply conditions which are apparently much stronger. In the present circumstances, we do have some limits on this, as in the examples mentioned above. The final answers are not clear, however.

Another way to think about the uniformity over all locations and scales in the uniform local coordinates property is that it is a condition which implies the existence of homeomorphic coordinates even after one "blows up" the space (in the Hausdorff or Gromov–Hausdorff senses) along any sequence of locations and scales in the space. With the uniform local coordinates property, the local coordinates could be "blown up" along with the space, with the uniform bounds for the moduli of continuity providing the equicontinuity and compactness needed to take limits of the coordinate mappings (after passing to suitable subsequences).

Instead of looking at uniform local coordinates in connection with *bilipschitz* equivalence with $\mathbf{R}^n$, one can consider *quasisymmetric* equivalence. Roughly speaking, a *quasisymmetric* mapping between two metric spaces is one that approximately preserves *relative* distances, in the same way that bilipschitz mappings approximately preserve actual distances. In other words, if one has three points x, y, and z in the domain of such a mapping, and if x is much closer to y than z is, then this should also be true for their images under a *quasisymmetric* mapping, even if the actual distances between the points might be changed a lot. See [TukV] for more details and information about quasisymmetric mappings. Two metric spaces are *quasisymmetrically equivalent* if there is a quasisymmetric mapping from one onto the other. As with bilipschitz mappings, compositions and inverses of quasisymmetric mappings remain quasisymmetric.

If a metric space admits a quasisymmetric parameterization from $\mathbf{R}^n$, then it also satisfies the condition of uniform local coordinates. This is true for nearly the same reason as for bilipschitz mappings; given a quasisymmetric mapping from $\mathbf{R}^n$ onto the metric space, one can get suitable local coordinates for the space from restrictions of the global mapping to indi-

vidual balls. There is a difference between this case and that of bilipschitz mappings, which is that one should allow some extra rescalings to compensate for the fact that distances are not approximately preserved. Specifically, a ball B in the metric space may be covered in a nice way by the image under a quasisymmetric mapping of a ball β in $\mathbf{R}^n$, but then there is no reason why the radii of B and β should be approximately the same. For a bilipschitz mapping, one would be able to choose β so that it has radius which is comparable to that of B. In the quasisymmetric case, one may not have that, but if one adds an extra rescaling on $\mathbf{R}^n$ (depending on the choices of balls), then one can still get local coordinates with the kind of uniform control on the (normalized) moduli of continuity as in the uniform local coordinates condition.

If a metric space has uniform local coordinates with respect to $\mathbf{R}^n$, and if these coordinates come from restrictions and then rescalings (on $\mathbf{R}^n$) of a single global parameterization, as in the preceding paragraph, then that parameterization does have to be quasisymmetric. This is an easy consequence of the definitions, and it is analogous to what happens in the bilipschitz case.

If a metric space admits uniform local coordinates from some $\mathbf{R}^n$, it still may not be true that it admits a quasisymmetric parameterization. This is trickier than before, and in particular one does not get examples simply by using snowflake metrics $|x - y|^\alpha$ on $\mathbf{R}^n$. Indeed, the identity mapping on $\mathbf{R}^n$ *is* quasisymmetric as a mapping from $\mathbf{R}^n$ with the standard metric to $\mathbf{R}^n$ with the snowflake metric $|x - y|^\alpha$, $0 < \alpha < 1$.

However, there are counterexamples, going back to results of Rickman and Väisälä. That is, these are spaces which have uniform local coordinates (and are even somewhat nicer than that), but which do not admit quasisymmetric parameterizations. Basically, these spaces are Cartesian products, where the individual factors behave nicely in their own right, and where the combination mixes different types of geometry. A basic example (which was the original one) is to take a product of a snowflake with a straight line. Quasisymmetric mappings try to treat different directions in a uniform manner, and in the end this does not work for parameterizations of these examples. See Lemma 4 in [Tuk], and also [Väi4] and [AleV1].

These examples occur already in dimension 2. They do not behave well in terms of measure, though, and for Ahlfors-regularity (of dimension 2) in particular. This is a basic part of the story; compare with [AleV1, Tuk, Väi4].

As usual, dimension 1 is special. There are results starting from more primitive conditions, and good characterizations for the existence of qua-

sisymmetric parameterizations, in fact. See Section 4 of [TukV].

On the other hand, in dimension 2, there are positive results about having global quasisymmetric parameterizations for a given space, and with bounds. For these results one would assume Ahlfors-regularity of dimension 2, and some modest additional conditions concerning the geometry and topology, conditions which are weaker than "uniform local coordinates". See [DaviS4, HeiKo1, Sem3]. For these results, it is important that the dimension be 2, and not larger, because of the way that they rely on the existence of conformal mappings. We shall say a bit more about this in Subsection C.8.

If we go now to dimension 3, then there are counterexamples, even with good behavior in terms of measure. That is, there are examples of subsets of Euclidean spaces which are Ahlfors-regular of dimension 3, admit uniform local coordinates with respect to $\mathbf{R}^3$, but are not quasisymmetrically-equivalent to $\mathbf{R}^3$. These examples are based on the decompositions of $\mathbf{R}^3$ associated to "Bing doubling" (as in [Bin1, Bin7, Dave2]), using geometric realizations as in Subsubsection C.7.1. This is discussed in [Sem7]. The absence of a quasisymmetric parameterization in this case is close to a result in [FreS], although the general setting in [FreS] is a bit different.

Concerning the space $(\mathbf{R}^3/W) \times \mathbf{R}$ considered before, equipped with a nice geometry as from Subsubsection C.7.1 and [Sem7], it is not clear (to my knowledge) whether *quasisymmetric* parameterizations from $\mathbf{R}^4$ exist or not. We saw earlier that *bilipschitz* mappings do not exist, because of the line in $(\mathbf{R}^3/W) \times \mathbf{R}$ which has to have Hausdorff dimension at least 2 after any homeomorphism from $(\mathbf{R}^3/W) \times \mathbf{R}$ onto $\mathbf{R}^4$. These considerations of Hausdorff measure do not by themselves rule out the existence of a quasisymmetric mapping, as they do for bilipschitz mappings. (Compare with [Väi3], for instance.)

Similar remarks apply to double-suspensions of homology spheres. In particular, it is not known (to my knowledge) whether or not quasisymmetric parameterizations exist for them.

To summarize a bit, we have seen in this subsubsection how decomposition spaces might be "standard" topologically, being equivalent to $\mathbf{R}^n$ as a topological space, and still lead to geometries which are tricky, and which have some interesting structure. We shall discuss another class of examples like this in Subsubsection C.7.4. These examples are more trivial topologically, more tame geometrically, but still nontrivial geometrically. (They admit simple quasisymmetric parameterizations, but no bilipschitz parameterizations.)

See [Sem7] for some other types of geometric structure which can arise naturally from decompositions. For instance, cellular decompositions whose quotients are not manifolds — such as Bing's dogbone space ([Bin2], Example 4 on p64-65 in [Dave2]) — give rise to spaces which do not have "uniform local coordinates", but which do have nice properties in other ways. For instance, one can make constructions so that local coordinates exist at all locations and scales, and with uniform bounds on how they are localized, but not for their moduli of continuity. This last is related to having Gromov–Hausdorff limits which are not manifolds, but are topologically given as a decomposition space (with a cellular decomposition), like Bing's dogbone space.

C.7.4 Examples that are even simpler topologically, but still nontrivial geometrically

Let us mention another class of examples, which one can also think of in terms of decompositions (although they are "trivial" in this respect). These examples are based on "Antoine's necklaces", which came up before, in Subsection C.1.

Antoine's necklaces are compact subsets of $\mathbf{R}^3$ which are homeomorphic to the usual middle-thirds Cantor set in the real line, but for which there is no global homeomorphism from $\mathbf{R}^3$ onto itself which maps these sets into subsets of a line. In dimension 2 this does not happen, as in Chapter 13 of [Moi].

The "wildness" of these sets is manifested in a simple fundamental-group property. Namely, the complement of these sets in $\mathbf{R}^3$ have nontrivial fundamental group, whereas this would not be true if there were a global homeomorphism from $\mathbf{R}^3$ to itself which would take one of these sets to a subset of the line. This last uses the fact that these sets are totally disconnected (i.e., to have simple-connectivity of the complement in $\mathbf{R}^3$ if the set were to lie in a line).

Antoine's necklaces are discussed in Chapter 18 of [Moi]. See also p71ff in [Dave2]. The basic construction for them can be described in terms of the same kind of "rules" as in Subsubsection C.7.1.

One starts with a solid torus T in $\mathbf{R}^3$. Inside this torus one embeds some more tori, which are disjoint, but which form a chain that is "linked" around the hole in the original torus. (See Figure 18.1 on p127 of [Moi], or Figure 9.9 on p71 of [Dave2].) In each of these smaller tori, one can embed another

collection of linking tori, in the same way as for the first solid torus.

One can repeat this indefinitely. In the limit, one gets a Cantor set, which is a necklace of Antoine.

Actually, we should be a little more precise here. In saying that we get a Cantor set in the limit, we are implicitly imagining that the diameters of the solid tori are going to 0 as one proceeds through the generations of the construction. This is easy to arrange, if one uses enough tori in the basic rule (linking around the original torus T). If one uses enough tori, then one can do this in such a way that all of the tori are *similar* to the initial torus T, i.e., can be given as images of T by mappings which are combinations of translations, rotations, and dilations.

One can also consider using a smaller number of tori, and where the embeddings of the new tori (in the original one) are allowed to have some stretching. (Compare with Figure 9.9 on p71 of [Dave2].) As an extreme case, the Whitehead continuum corresponds essentially to the same construction as Antoine's necklaces, except that only 1 embedded torus is used in the linking in the original torus T. (See Figure 9.7 on p68 of [Dave2].) For this one definitely needs a fair amount of stretching. In Bing doubling, one uses two solid tori, embedded and linked around the hole in T (Figure 3 on p357 of [Bin1], Figure 9.1 on p63 in [Dave2]). One again needs some stretching, but not as much for the Whitehead continuum.

These cases are different from the standard ones for Antoine's necklaces, because when one iterates the basic rule in a straightforward way, the components that one gets at the nth generation do not have diameters tending to 0. For the Whitehead continuum, this is simply unavoidable, and reflects the way that the components are clasped, each one by itself, around the "hole" of the original torus T. In the case of Bing doubling, one can rearrange the embeddings at later generations in such a way that the diameters do tend to 0, even if this might not be true for naive iterations. This was proved by Bing, in [Bin1]. (See also p69-70 of [Dave2], and [Bin7].)

Let us imagine that we are using enough small solid tori in the linking around T, as in standard constructions for Antoine's necklaces, so that it is clear that the diameters of the components of the sets obtained by repeating the process do go to 0 (i.e., without having to make special rearrangements, or anything like that, as in Bing doubling). In other words, in the limit, one gets a Cantor set in $\mathbf{R}^3$, as above. Let us call this Cantor set A.

This Cantor set A is wild, in the sense that its complement in $\mathbf{R}^3$ is not simply-connected, and there is no global homeomorphism from $\mathbf{R}^3$ to itself

which takes A to a subset of a line. At the level of decompositions, however, there is not much going on here.

Normally, to get a decomposition from a process like this, one takes the connected components obtained in the limit of the process, together with sets with one element for the rest of the points in $\mathbf{R}^3$ (or $\mathbf{R}^n$, as the case may be). This was described before, in Subsubsection C.7.1. (See also the discussion at the beginning of Section 9 of [Dave2], on p61, concerning the notion of a *defining sequence*.) In the construction that we are considering here, all of the connected components in the end contain only one element each, because of the way that the diameters of the components in finite stages of the "defining sequence" converge to 0.

In other words, the decomposition that occurs here is automatically "trivial", consisting of one-element sets, one for each point in $\mathbf{R}^3$. Taking the quotient does not do anything, and the decomposition space is just $\mathbf{R}^3$ again.

There is nothing too complicated about this. It is just something to say explicitly, for the record, so to speak, to be clear about it, especially since it is a situation to which one might normally pay little attention, for being degenerate. (See also the text at the beginning of p71 of [Dave2], about this kind of defining sequence and decomposition.)

While there is nothing going on at the level of the decompositions *topologically*, this is not the case geometrically! One can think of this in the same way as in Subsubsection C.7.1, for making geometric representations of decomposition spaces, with metrics and self-similarity properties for them. In the present situation, one can also work more directly at the level of $\mathbf{R}^3$ itself, to get geometries like this.

Here is the basic point. Imagine deforming the geometry of $\mathbf{R}^3$, at the level of infinitesimal measurements of distance, as with Riemannian metrics. In the general idea of a decomposition space, one can shrink sets in some $\mathbf{R}^n$ which have nonzero diameter to single points. In the present setting, our basic components already *are* single points, and so there is nothing to do to them. However, one can still shrink the *geometry* around these points in $\mathbf{R}^3$.

In technical terms, one can think of deforming the geometry of $\mathbf{R}^3$, by multiplying its standard Riemannian metric by a function. One can take this function to be positive and regular away from the Antoine's necklace, and then vanish on the necklace itself. For instance, one could take the function to be a positive power of the distance to the necklace A, so that the

Riemannian metric can be written as follows:

$$ds^2 = \mathrm{dist}(x, A)^\alpha \, dx^2. \tag{C.24}$$

This type of metric is discussed in some detail in [Sem8] (although in a slightly different form). The geometry of $\mathbf{R}^3$ with this kind of metric behaves a lot like standard Euclidean geometry. One still has basic properties like Ahlfors-regularity of dimension 3, and Sobolev, Poincaré, and isoperimetric inequalities in the new geometry. See [Sem8]. (This kind of deformation is special case of the broader class in [DaviS1] and [Sem5, Sem10].)

However, with suitable choices of parameters, the metric space that one gets in this way is *not* bilipschitz equivalent to $\mathbf{R}^3$ with the standard Euclidean metric. This is because the shrinking of distances around the necklace A can lead to A having Hausdorff dimension less than 1 in the new metric. On the other hand, $\mathbf{R}^3 \backslash A$ is not simply-connected, and this means that the geometry which has been constructed cannot be bilipschitz equivalent to the standard geometry on $\mathbf{R}^3$, because of Theorem C.19 in Subsection C.5. See [Sem8] for more information. (Concerning the "choice of parameters" here, the main point is to have enough shrinking of distances around A to get the Hausdorff dimension to be less than 1, or something like that. The amount of shrinking needed depends on some of the choices involved in producing the necklace. By using Antoine's necklaces which are sufficiently "thin", it is enough to employ arbitrarily small powers α of the distance to the necklace in (C.24) to get enough shrinking of the metric around the necklace. In any case, one is always free to take the power α to be larger.)

Let us emphasize that in making this kind of construction, the conclusion is that the metric space that one gets is not bilipschitz-equivalent to $\mathbf{R}^3$ through *any* homeomorphism between the two spaces. It is easy to make deformations of the geometry so that the new metric seems to be much different from the old one in the given coordinates, but for which this is not really the case if one is allowed to make a change of variables. For instance, one might deform the standard Euclidean Riemannian metric on $\mathbf{R}^3$ by multiplying it by a function that vanishes at a point, like a positive power of the distance to that point. Explicitly, this means

$$d\tilde{s}^2 = |x - p|^\beta \, dx^2, \tag{C.25}$$

where $\beta > 0$. In the standard coordinates, this metric and the ordinary one look quite different. However, for this particular type of deformation (as in

(C.25)), the two metrics are bilipschitz equivalent, if one allows a nontrivial change of variables. Specifically, one can use changes of variables of the form

$$f(x) = p + |x - p|^{\beta/2}(x - p). \tag{C.26}$$

This is not hard to verify.

As a more complicated version of this, one can also make deformations of the standard geometry on $\mathbf{R}^3$ of the form

$$d\hat{s}^2 = \mathrm{dist}(x, K)^\gamma \, dx^2, \tag{C.27}$$

where $\gamma > 0$ and K is a self-similar Cantor set in $\mathbf{R}^3$ which is *not* wild. In this case one can again get geometries which may look different from the usual one in the given coordinates, but for which there are changes of variables which give a bilipschitz equivalence with the standard metric. Compare with Remark 5.28 on p390 of [Sem8].

When one makes deformations based on Antoine's necklaces, as above, the linking that goes on can ensure that there is no bilipschitz equivalence between $\mathbf{R}^3$ with the new geometry and $\mathbf{R}^3$ with the standard geometry. In fact, there will not be a homeomorphism which is Lipschitz (from the new geometry to the standard one), without asking for bilipschitzness. With suitable choices of parameters, it can be impossible to have a homeomorphism like this which is even Hölder continuous of an arbitrary exponent $\delta > 0$, given in advance.

Under the conditions in [Sem8], the identity mapping itself on $\mathbf{R}^3$ always gives a homeomorphism which is Hölder continuous with *some* positive exponent. In fact, it is also *quasisymmetric*, in the sense of [TukV]. Thus, here one gets examples of spaces which are quasisymmetrically equivalent to a Euclidean space, and which are Ahlfors-regular of the correct dimension, but which are not bilipschitz equivalent to a Euclidean space. (Compare with Subsubsection C.7.3.)

C.8 Geometric and analytic results about the existence of good coordinates

In Subsection C.2, we considered the question of whether a nonempty contractable open set in $\mathbf{R}^n$ is homeomorphic to the standard open unit ball in $\mathbf{R}^n$. When $n = 2$ this is true, and it is a standard result in topology.

One can establish this result in 2 dimensions analytically via the *Riemann Mapping Theorem*. This theorem gives the existence of a *conformal* mapping from the unit disk in $\mathbf{R}^2$ onto any nonempty simply-connected open set in $\mathbf{R}^2$ which is not all of $\mathbf{R}^2$. See Chapter 6 of [Ahl1], for instance.

The Riemann Mapping Theorem is of course very important for many aspects of analysis and geometric function theory in $\mathbf{R}^2 \cong \mathbf{C}$, but it also does a lot at a less special level. From it one not only obtains homeomorphisms from the unit disk in $\mathbf{R}^2$ onto any nonempty simply-connected proper open subset of $\mathbf{R}^2$, but one gets a way of choosing such homeomorphisms which is fairly canonical. In particular, Riemann mappings are unique modulo a three real-dimensional group of automorphisms of the unit disk (which can be avoided through suitable normalizations), and there are results about the dependence of Riemann mappings on the domains being parameterized.

By comparison, one might try to imagine doing such things without the Riemann mapping, or in other contexts where it is not available. In this regard, see [Hat1, Hat2, Lau, RanS], concerning related matters in higher dimensions.

Another fact in dimension 2 is that any smooth Riemannian metric on the 2-sphere $\mathbf{S}^2$ is conformally-equivalent to the standard metric. In other words, if g is a smooth Riemannian metric on $\mathbf{S}^2$, and if g_0 denotes the standard metric, then there is a diffeomorphism from $\mathbf{S}^2$ onto $\mathbf{S}^2$ which converts g into a metric of the form $\lambda \, g_0$, where λ is a smooth positive function on $\mathbf{S}^2$. There are also local and other versions of this fact, but for the moment let us stick to this formulation.

One way to try to use this theorem is as follows. Suppose that one has a 2-dimensional space which behaves roughly like a 2-dimensional Euclidean space (or sphere) in some ways, and one would like to know whether it can be realized as nearly-Euclidean in more definite ways, through a parameterization which respects the geometry. Let us assume for simplicity that our space is given to us as $\mathbf{S}^2$ with a smooth Riemannian metric g, but without bounds for the smoothness of g. One can then get a conformal diffeomorphism $f : (\mathbf{S}^2, g_0) \to (\mathbf{S}^2, g)$, as in the result mentioned in the preceding paragraph. A priori the behavior of this mapping could be pretty complicated, and one might not know much about it at definite scales. It would be nice to have some bounds for the behavior of f, in terms of simple geometric properties of $(\mathbf{S}^2, g)$.

Some results of this type are given in [DaviS4, HeiKo1, Sem3]. Specifically, general conditions are given in [DaviS4, HeiKo1, Sem3] under which a

conformal equivalence $f : (\mathbf{S}^2, g_0) \to (\mathbf{S}^2, g)$ actually gives a *quasisymmetric* mapping (as in [TukV]), with uniform bounds for the quasisymmetry condition. In other words, these results have the effect of giving uniform bounds for the behavior of f at any location or scale, under suitable conditions on the initial space $(\mathbf{S}^2, g)$, and using the conformality of f.

Another use of conformal mappings of an analogous nature is given in [MülŠ]. There the assumptions on the space involved more smoothness — an integral condition on (principal) curvatures, for a surface in some $\mathbf{R}^n$ — and the conclusions are also stronger, concerning bilipschitz coordinates. This gave a new approach to results in [Tor1]. See also [Tor2], and the recent and quite different method in [Fu].

More precisely, [MülŠ] works with conformal mappings, while [Tor1, Tor2] and [Fu] obtain bilipschitz coordinates by quite different means. In the context of [DaviS4, HeiKo1, Sem3], no other method for getting quasisymmetric or other coordinates with geometric estimates (under similar conditions) is known, at least to my knowledge.

One might keep in mind that conformality is defined in infinitesimal terms, through the differential of f. To go from infinitesimal or very small-scale behavior to estimates at larger scales, one in effect tries to "integrate" the information that one has.

This is a very classical subject for conformal and quasiconformal mappings. A priori, it is rather tricky, because one is not given any information about the conformal factors (like the function λ before). Thus one cannot "integrate" directly in a conventional sense. One of the basic methods is that of "extremal length", which deals with the balance between length and area. At any rate, methods like these are highly nonlinear. See [Ahl2, Ahl3, LehV, Väi1] for more information.

What would happen if one attempted analogous enterprises in higher dimensions? One can begin in the same manner as before. Let n be an integer greater than or equal to 2, and suppose (as a basic scenario) that one has a smooth Riemannian metric g on $\mathbf{S}^n$. Let g_0 denote the standard metric on $\mathbf{S}^n$. One might like to know that $(\mathbf{S}^n, g)$ can be parameterized by $(\mathbf{S}^n, g_0)$ through a mapping with reasonable properties, and with suitable bounds, under some (hopefully modest) geometric conditions on $(\mathbf{S}^n, g)$. Here, as before, the smoothness of g should be taken in the character of an a priori assumption. One would seek uniform bounds that do not depend on this in a quantitative way. (The bounds would depend on constants in the geometric conditions on $(\mathbf{S}^n, g)$.)

If one has a mapping $f : (\mathbf{S}^n, g_0) \to (\mathbf{S}^n, g)$ which is conformal, or which is quasiconformal (with a bound for its dilatation), then [HeiKo1] provides some natural hypotheses on $(\mathbf{S}^n, g)$ under which one can establish that f is quasisymmetric, and with bounds. In other words, this works for all $n \geq 2$, and not just $n = 2$, as above. See [HeiKo2, HeiKo2] for further results along these lines.

However, when $n > 2$, there are no general results about *existence* of conformal parameterizations for a given space, or quasiconformal parameterizations with uniform bounds for the dilatation. It is simply not true that arbitrary smooth Riemannian metrics admit conformal coordinates, even locally, as they do when $n = 2$. Quasiconformal coordinates automatically exist for reasonably-nice metrics, but with the quasiconformal dilatation depending on the metric or on the size of the region being parameterized in a strong way. The issue would be to avoid or reduce that.

One can easily see that the problem is highly overdetermined, in the following sense. A general Riemannian metric in n dimensions is described (locally, say) by $n(n+1)/2$ real-valued functions of n variables. A conformal deformation of the standard metric is defined by 1 real-valued function of n real variables, i.e., for the conformal factor. A general diffeomorphism in n dimensions is described by n real-valued functions of n variables. Thus, allowing for general changes of variables, the metrics which are conformally-equivalent to the standard metric are described by $n+1$ real-valued functions of n real variables. When $n = 2$, this is equal to $n(n+1)/2$, but for $n > 2$ one has that $n(n+1)/2 > n+1$.

In fact, one knows that in dimension 3 there are numerous examples of spaces which satisfy geometric conditions analogous to ones that work in dimension 2, but which do not admit quasisymmetric parameterizations. There are also different levels of structure which occur in dimension 3, between basic geometric properties and having quasisymmetric parameterizations, and which would come together in dimension 2. See [Sem7]. Parts of this are reviewed or discussed in Subsection C.7, especially Subsubsection C.7.3.

Thus, not only does the method based on conformal and quasiconformal mappings not work in higher dimensions, but some of the basic results that one might hope to get or expect simply are not true, by examples which are pretty concrete.

These examples can be viewed as geometrizations of classical examples from geometric topology, and they are based on practically the same principles. They are not especially strange or pathological or anything like that,

but have a lot of nice properties. They reflect basic phenomena that occur.

This is all pretty neat! One has kinds of "parallel tracks", with geometric topology on one side, and aspects of geometry and analysis on the other. A priori, these two tracks can exist independently, even if there are ways in which each can be involved in the other.

Each of these two tracks has special features in low dimensions. This concerns the existence of homeomorphisms with certain properties, for instance. Each has statements and results and machinery which make sense for the given track, and not for the other side, even if there are also some overlaps (as with applications of Riemann mappings).

Each of these two tracks also starts running into trouble in higher dimensions, and at about the same time! We have seen a number of instances of this by now, in this subsection, in this appendix more broadly, and also Section 3. The kinds of trouble that they encounter can be rather different a priori (such as localized fundamental group conditions, versus behavior of partial differential equations), even if there is again significant overlap between them. In particular, this concerns the existence of homeomorphisms with good properties.

C.8.1 Special coordinates that one might consider in other dimensions

We have already discussed a number of basic topological phenomena in this appendix. Let us now briefly consider a couple of things that one might try in higher dimensions on the side of geometry and analysis, in similar veins as above.

One basic approach would be to try to find and use mappings which minimize some kind of "energy". As before, one can consider smooth metrics on smooth manifolds (like $\mathbf{S}^n$), and try to get parameterizations with uniform bounds on their behavior, under modest conditions on the geometry of the spaces. (One can also try to work directly with spaces and metric that are not smooth.)

A very standard energy functional to consider would be the L^2 norm of the differential, as with harmonic mappings. In dimension 2, conformal mappings can be placed in this framework. One can also consider energy functionals based on L^p norms of differentials of mappings. This is more complicated in terms of the differential equations that come up, but it can have other advantages. The choice of p as the dimension n has some particularly nice

features, for having the energy functional cooperate with the geometry (and analysis). (This is one of the ways that $n = 2$ is special; for this one can have both $p = n$ and $p = 2$ at the same time!) In particular, the energy becomes invariant under conformal changes in the metric when p is equal to the dimension.

In elasticity theory, one considers more elaborate energy functionals as well. For instance, these might include integral norms of the *inverse* of the Jacobian of the mapping, in addition to L^p norms of the differentials. In other words, the functional can try to limit both the way that the differential becomes large and small, so that it takes into account both stretchings and compressions.

In any case, although there is a lot of work concerning existence and behavior of minimizers for functionals like these, I do not really know of results in dimensions $n \geq 3$ where they can be used to obtain well-behaved parameterizations of spaces, with bounds, under modest or general geometric conditions. This is especially true in comparison with what one can get in dimension 2, as discussed before.

This is a bit unfortunate, compared with the way that the normal form of conformal mappings can be so useful in dimension 2. On the other hand, perhaps someone will find ways of using such variational problems for geometric questions like these some day, or will find some kind of special structure connected to them. In this regard, one might bear in mind issues related to mappings with *branching*, as in Appendix B. We shall say a bit more about this later in this subsection.

One might also keep in mind the existence of spaces with good properties, but not good parameterizations, as mentioned earlier (and discussed in Subsubsections C.7.3 and C.7.4, and in [Sem7, Sem8]).

In dimension 3, there is another kind of special structure that one might consider. Namely, instead of metrics which are conformal deformations of the standard Euclidean metric, let us consider metrics $g = g_{i,j}$ for which only the *diagonal* entries $g_{i,i}$ are nonzero.

In this case the diagonal entries are allowed to vary independently. For conformal deformations of the standard Euclidean metric, the off-diagonal entries are zero, and the diagonal entries are all equal.

In dimension 3, the problem of making a change of variables to put a given metric into diagonal form like this is "determined", in the same way as for conformal deformations of Euclidean metrics in dimension 2. Specifically, one can compute as follows. A general Riemannian metric is described by

$n(n+1)/2$ real-valued functions of n variables, which means 6 real-valued functions of 3 real variables in dimension 3. Metrics with only diagonal nonzero entries are defined by 3 real-valued functions of 3-real variables, and changes of variables are given by 3 real-valued functions of 3 real variables as well. Thus, allowing for changes of variables, the metrics that can be reduced to diagonal metrics can be described by 6 real-valued functions of 3 real-variables, which is the same as for the total family of Riemannian metrics in this dimension. In dimensions greater than or equal to 4, this would not work, and there would again be too many Riemannian metrics in general compared to diagonal metrics and ways of reducing to them via changes of variables.

Of course this is just an informal "dimension" count, and not a justification for being able to put metrics into diagonal form in dimension 3. (One should also be careful that there is not significant overlap between changes of variables and diagonal metrics, i.e., so that there was no "overcounting" for the combination of them.) However, it does turn out that one can put metrics in diagonal form (in dimension 3), at least locally. This was established in [DeTY] in the case of smooth metrics. There were earlier results in the real-analytic category. (See [DeTY] for more information.)

However, this type of "normal form" does not seem to be as useful for the present type of issue as conformal parameterizations are. As in the case of conformal coordinates, part of the problem is that even if one has such a normal form, one does not a priori know anything about the behavior of the diagonal entries of the metric in this normal form. One would need methods of getting estimates without this information, and only the nature of the normal form. In the context of conformal mappings, one has extremal lengths, conformal capacities, and other conformal and quasiconformal invariants and quasi-invariants. For diagonal metrics, it is not clear what one might do.

A related point is that the analysis of the partial differential equations which permits one to put smooth Riemannian metrics in dimension 3 into diagonal form is roughly "hyperbolic", in the same way that the corresponding differential equations for conformal coordinates in dimension 2 are elliptic. See [DeTY]. This is closely connected to the kind of stability that one has for conformal mappings, and the possibilities for having estimates for them under mild or primitive geometric conditions.

In a way this is all "just fair", and nicely so. With diagonal metrics one does have something analogous to conformal coordinates in dimension 3. On the other hand, this analogue behaves differently in fundamental ways,

including estimates. This is compatible with other aspects of the story as a whole, like the topological and geometric examples that one has in dimension 3 (where homeomorphisms may not exist, with the properties that one might otherwise hope for).

In any event, this illustrates how analytic and geometric methods seem to behave rather differently in dimensions 3 and higher, compared to the special structure and phenomena which occur in dimension 2. This is somewhat remarkable in analogy with topological phenomena, which have similar differences between dimensions. With the topology there are both some crossings and overlaps with geometry and analysis, and much that is separate or independent.

On the side of geometry and analysis, let us also note that there are some other special features in low dimensions that we have not mentioned. As a basic example, the large amount of flexibility that one has in making conformal mappings in dimension 2 leads to some possibilities in dimension 3 that are not available in higher dimensions. That is, the large freedom that one has in dimension 2 can sometimes permit one to make more limited constructions in dimension 3, e.g., by starting with submanifolds of dimension 2, and working from there (with extensions, gluings, etc.) These possibilities in dimension 3 can be much more restricted than in dimension 2, but having them at all can be significantly more than what happens in higher dimensions.

We should also make clear that if one allows mappings with *branching*, as in Appendix B, then a number of things can change. Some topological difficulties could go away or be ameliorated, as has been indicated before (and in Appendix B). For instance, the branching can unwind obstructions or problems with localized fundamental groups (in complements of points or other sets). There are many basic examples of this, as in Appendix B, and the constructions in [HeiR1, HeiR2]. Ideas of Sullivan [Sul2, Sul3] are also important in this regard (and as mentioned in Appendix B).

General pictures for mappings with branching, including existence and good behavior, have yet to be fully explored or understood. The Alexander argument described in Appendix B, the constructions of Heinonen and Rickman [HeiR1, HeiR2], classical work on quasiregular mappings (as in [Res, Ric1]), and the work of Sullivan [Sul2, Sul3], seem to indicate many promising possibilities and directions.

One can perhaps use variational problems in these regards as well.

Concerning variational problems, one might also keep in mind the approaches of [DaviS9, DaviS11] (and some earlier ideas of Morel and Solimini

[MoreS]). For these one does not necessarily work directly with mappings or potential parameterizations of sets, and in particular one may allow sets themselves to be variables in the minimization (rather than mappings between fixed spaces). This broader range can make it easier for the minimizations to lead to useful conclusions about geometric structure and complexity, under natural and modest conditions. In particular, one can get substantial "partial parameterizations", as with uniform rectifiability conditions.

These approaches are also nicely compatible with the trouble that one knows can occur, related to topology and homeomorphisms (and in geometrically moderate situations, as in Subsections C.7 and C.7.4, and [Sem7, Sem8]).

Finally, while we have mentioned a lot about the special phenomena that can occur in dimension 2, and what happens in higher dimensions, we should also not forget about dimension 1. This is even more special than dimension 2. This is a familiar theme in geometric topology, for the ways that one can recognize and parameterize curves. In geometry and analysis, one can look for parameterizations with bounds, and these are often constructible.

A fundamental point along the lines is the ability to make parameterizations by arclength, for curves of locally finite length. More generally, one can use parameterizations adapted to other measures (rather than length), when they are around.

Arclength parameterizations provide a very robust and useful way for obtaining parameterizations in dimension 1 with good behavior and bounds. In dimension 1, simple conditions in terms of mass can often be immediately "integrated" to get well-behaved parameterizations, in ways that are not available (or do not work nearly as well) in higher dimensions, even in dimension 2.

To put the matter in more concrete terms, in dimension 1 one can often make parameterizations, or approximate parameterizations, simply by ordering points in a good way. This does not work in higher dimensions. Once one has the ordering, one can regularize the geometry by parameterizing according to arclength, or some other measure (as appropriate).

For another version of this, in connection with quasisymmetric mappings, see Section 4 of [TukV].

In differential-geometric language, one might say that dimension 1 is special for the way that one can make isometries between spaces, through arclength parameterizations. This no longer works in dimension 2, but one has conformal coordinates there. Neither of these are generally available in

higher dimensions. In higher dimensions one has less special structure for getting the existence in general of well-behaved parameterizations, and then the kinds and ranges of geometric and topological phenomena which can exist open up in a large way.

C.9 Nonlinear similarity: Another class of examples

A very nice and concrete situation in which issues of existence and behavior of homeomorphisms can come up is that of "nonlinear similarity". Specifically, it is possible to have linear mappings A, B on $\mathbf{R}^n$ which are conjugate to each other by homeomorphisms from $\mathbf{R}^n$ onto itself — i.e., $B = h \circ A \circ h^{-1}$, where h is a homeomorphism of $\mathbf{R}^n$ onto itself — and which are *not* conjugate by *linear* mappings!

Examples of this were given in [CapS2]. For related matters, including other examples and conditions under which one can deduce linear equivalence, see [CapS1, CapS3, CapS4, CapS5, CapS+, CapS∗, HamP1, HamP2, HsiP1, HsiP2, KuiR, MadR1, MadR2, Mio, Rha1, Rha2, RotW, Wei1, Wei2, Wilk].

Note that if one has a conjugation of linear mappings A and B by a *diffeomorphism* h on $\mathbf{R}^n$, then one can derive the existence of a linear conjugation from this. This comes from passing to the differential of the diffeomorphism at the origin.

Thus, when a linear conjugation does not exist, but a homeomorphic conjugation does, then the homeomorphism cannot be smooth at all points in $\mathbf{R}^n$, or even at just the origin. One might wonder then about the kinds of processes and regularity that might be entailed in the homeomorphisms that provide the conjugation. In this regard, see [CapS4, RotW, Wei1].

D Doing pretty well with spaces which may not have nice coordinates

If one has a topological or metric space (or whatever) which has nice coordinates, then that can be pretty good.

However, there is a lot that one can do without having coordinates. As in Section 3 and Appendix C, there are many situations in which homeomorphic coordinates might not be available, or might be available only in irregular forms, or forms with large complexity. Even if piecewise-linear coordinates

exist, for instance, it may not be very nice if they have enormous complexity, as in Section 3.

Or, as in Section 3, local coordinates might exist, but one might not have an algorithmic way to know this. Similarly, coordinates might exist, but it may not be so easy to find them.

As a brief digression, let us mention some positive results about situations in which coordinates exist, but are not as regular as one might like, at least not at first. Consider the case of topological manifolds, which admit homeomorphic local coordinates, but for which there may not be a compatible piecewise-linear or smooth structure. Homeomorphic coordinates are not suitable for many basic forms of analysis in which one might be interested, e.g., involving differential operators. However, there is a famous theorem of Sullivan [Sul1], to the effect that topological manifolds of dimension ≥ 5 admit unique quasiconformal and Lipschitz structures (for which one then has quasiconformal and bilipschitz coordinates). (In dimensions less than or equal to 3, unique piecewise-linear and smooth structures always exist for topological manifolds, by more classical results. In dimension 4, quasiconformal and Lipschitz structures may not exist for topological manifolds, or be unique. See [DonS]. Concerning smooth structures in dimension 4, see [DonK, FreQ]. Some brief surveys pertaining to different structures on manifolds, and in general dimensions, are given in Section 8 in [FreQ], and the "Epilogue" in [MilS].)

Thus, with Sullivan's theorem, one has the possibility of improving the structure in a way that does make tools of analysis feasible. Some references related to this include [ConST1, ConST2, DonS, RosW, Sul1, SulT, Tel1, Tel2].

Some aspects of working on spaces without good coordinates came up in Section 4. One could also consider "higher-order" versions of this, along the lines of differential forms. We shall not pursue this here, but for a clear and simple version of this, one can look at the case of Euclidean spaces with the geometry deformed through a metric doubling measure (as in [DaviS1, Sem5, Sem10]). Some points about this are explained in [Sem10], beginning near the bottom of p427. (Compare also with [Sem8], concerning the possible behavior of Euclidean spaces with geometry deformed by metric doubling measures.)

In this appendix, we shall focus more on traditional objects from algebraic topology, like homology and cohomology groups. In this setting, there is a lot of structure around, concerning spaces which might be approximately like

manifolds, but not quite manifolds.

Let us begin with some basic conditions. Let M be a topological space which is compact, Hausdorff, and metrizable. We shall assume that M has finite topological dimension, in the sense of [HurW]. In these circumstances, this is equivalent to saying that M is homeomorphic to a subset of some $\mathbf{R}^n$. (See [HurW].)

For simplicity, let us imagine that M simply *is* a compact subset of some $\mathbf{R}^n$. It is also convenient to ask that M be *locally contractable*. This means (as in Subsubsection C.7.2) that for each point $p \in M$, and each neighborhood U of p in M, there is a smaller neighborhood $V \subseteq U$ of p in M such that V can be contracted to a point in M.

As a class of examples, finite polyhedra are locally contractable. Finite polyhedra make a nice special case to consider throughout this appendix, and we shall return to it several times.

For another class of examples, one has cell-like quotients of topological manifolds (and some locally contractable spaces more generally), at least when the quotient spaces have finite topological dimension. See Corollary 12B on p129 of [Dave2]. (Cell-like quotients were discussed somewhat in Appendix C, Subsections C.4 and C.6 in particular. Some concrete instances of cell-like quotients are mentioned in these subsections, and [Dave2] provides more examples and information.)

Although we shall mostly not emphasize metric structures or quantitative aspects in the appendix, let us mention that cell-like quotients like these often have natural and nice geometries, as indicated in Subsection C.7. These geometries are quite different from those of finite polyhedra, but they can also have some analogous properties. In particular, there can be forms of self-similarity or scale-invariant boundedness of the geometry, and these can be analogous to local conical structure in polyhedra in their effects. They are not as strong or special, and they are also more flexible. In any case, local contractability (and conditions like local linear contractability, in some geometric settings) is a basic property to perhaps have.

As a general fact about local contractability, let us note the following.

Proposition D.1 *Let M be a compact subset of some $\mathbf{R}^n$. Then M is locally contractable if and only if there is a set $V \subseteq \mathbf{R}^n$ which contains M in its interior, and a continuous mapping $r : V \to M$ which is a* retract, *i.e., $r(w) = w$ for all $w \in V$.*

This is a fairly standard observation. The "if" part is an easy consequence of the local contractability of $\mathbf{R}^n$ (through linear mappings). I.e., to get local contractions inside of M, one makes standard linear contractions in $\mathbf{R}^n$, which normally do not stay inside M, and then one applies the retraction to keep the contractions inside $\mathbf{R}^n$.

For the converse, one can begin by defining r on a discrete and reasonably-thick set of points outside M, but near M. For such a point w, one could choose $r(w) \in M$ so that it lies as close to w as possible (among points in M), or is at least approximately like this. To fill in r in the areas around these discrete points, one can make extensions first to edges, then 2-dimensional faces, and so on, up to dimension n. To make these extensions, one uses local contractability of M. It is also important that the local extensions do not go to far from the selections already made, so that $r : V \to M$ will be continuous in the end, and this one can also get from the local contractability.

The notion of "Whitney decompositions", as in Chapter VI of [Ste1], is helpful for this kind of argument. It gives a way of decomposing $\mathbf{R}^n \backslash M$ into cubes with disjoint interiors, and some other useful properties. (In particular, this kind of decomposition can be helpful for keeping track of bounds, if one should wish to do so.) One can use the vertices of these cubes for the discrete set in the complement of M mentioned above.

See also [Dave2] for a proof of Proposition D.1, especially p117ff.

Let us return now to the general story. Suppose that M is a compact subset of $\mathbf{R}^n$, and that M is locally contractable. Let $r : V \to M$ be a continuous retraction on M, as in Proposition D.1. Thus V contains M in its interior. By replacing V by a slightly smaller subset, if necessary, we may assume that V is compact, and in fact that it is a finite union of dyadic cubes in $\mathbf{R}^n$. (A *dyadic cube* in $\mathbf{R}^n$ is one which can be represented as a Cartesian product of intervals $[j_i \, 2^{-k}, (j_i + 1) \, 2^{-k}]$, $i = 1, 2, \ldots, n$, where the j_i's and k are integers.)

This type of choice for V is convenient for having nice properties in terms of homology and cohomology. In particular, V is then a finite complex. The inclusion of M into V, and the mapping $r : V \to M$, induce mappings between the homology and cohomology of M and V. If $\iota : M \to V$ denotes the mapping coming from inclusion, then $r \circ \iota : M \to M$ is the identity mapping, and thus it induces the identity mapping on the homology and cohomology of M. Using this, one can see that the mapping from the homology of M into the homology of V induced by ι is an injection (in addition to being a group homomorphism, as usual), and that the mapping from the homology of V to

the homology of M induced by r is a surjection. This follows from standard properties of homology and mappings, as in [Mas]. Similarly, r induces a mapping from cohomology of M to cohomology of V which is injective, and ι induces a mapping from cohomology of V into cohomology of M which is surjective.

This provides a simple way in which the algebraic topology of M can be "bounded", under the type of assumptions on M that we are making. (There are more refined things that one can also do, but we shall not worry about this here.) Local contractability, and the existence of a retraction as in Proposition D.1, are also nice for making it clear and easy to work with continuous mappings into M. In particular, one can get continuous mappings into M from continuous mappings into V, when one has a retraction $r : V \to M$, as above. This is as opposed to standard examples like the closure of the graph of $\sin(1/x)$, $x \in [-1, 1] \backslash \{0\}$. (This set is connected but not arcwise connected.)

Now let us consider the following stronger conditions on M.

Definition D.2 (Generalized k-Manifolds) *Let M be a compact subset of $\mathbf{R}^n$ which is locally contractable. Then M is a* generalized k-manifold *if for every point $z \in M$, the relative homology $H_j(M, M \backslash \{z\})$ is the same (up to isomorphism) as the relative homology of $H_j(\mathbf{R}^k, \mathbf{R}^k \backslash \{0\})$ for each j. (In other words, $H_j(M, M \backslash \{z\})$ should be zero when $j \neq k$, and isomorphic to $\mathbf{Z}$ when $j = k$.)*

We are implicitly working with homology defined over the integers here, and there are analogous notions with respect to other coefficient groups (like rational numbers, for instance). One may also wish to use weaker conditions than local contractability (as in [Bred2, Wild]). There are other natural variations for this concept.

If M is a finite polyhedron, then the property of being a generalized manifold is equivalent to asking that the links of M be homology spheres (i.e., have the same homology as standard spheres, up to isomorphism) of the right dimension. Such a polyhedron may not be a topological manifold (in dimensions greater than or equal to 4), because the codimension-1 links may not be simply-connected. (This is closely related to some of the topics of Appendix C, and the condition (C.9) in Subsection C.3 in particular.)

Another class of examples comes from taking quotients of compact topological manifolds by cell-like decompositions (Subsections C.4 and C.6), at

least when the quotient space has finite topological dimension. See Corollary 1A on p191 of [Dave2] (and Corollary 12B on p129 there), and compare also with Theorem 16.33 on p389 of [Bred2], and [Fer4]. As in Appendix C (especially Subsections C.4 and C.6) and the references therein, such spaces are *not* always topological manifolds themselves, in dimensions 3 and higher. For a concrete example of this, one can take the standard 3-sphere, and collapse a copy of the Fox–Artin (wild) arc to a point. (Compare with [Fer4].) One could also collapse a Whitehead continuum to a point (where Whitehead continua are as in Subsection C.2), as in Subsection C.4. As in Appendix C, the fact that these spaces are not manifolds can be seen in the nontriviality of localized fundamental groups, i.e., fundamental groups of the complement of the distinguished point (corresponding to the Fox–Artin curve or Whitehead continuum), localized at that point. This is analogous to the possibility of having codimension-1 links in polyhedra which are not simply-connected, as in the previous paragraph.

As usual, dimensions 1 and 2 are special for generalized manifolds, which are then topological manifolds. See [Wild], Theorem 16.32 on p388 of [Bred2], and the introduction to [Fer4].

For more on ways that generalized manifolds can arise, see [Bor2, Bred2, Bry+, Bry∗, Dave2, Fer4, Wei2] (and the references therein). A related topic is the "recognition problem", for determining when a topological space is a topological manifold. This is also closely connected to the questions in Appendix C. Some references for this include [Bry+, Bry∗, Can1, Can2, Can3, Dave2, Edw2, Fer4, Wei2].

What are some properties of generalized manifolds? In what ways might they be like manifolds? In particular, how might they be different from compact sets in $\mathbf{R}^n$ which are locally contractable in general?

A fundamental point is that Poincaré duality (and other duality theorems for manifolds) also work for generalized manifolds. See [Bor1, Bor2, Bred2, Wild] and p277-278 of [Spa]. This is pretty good, since Poincaré duality is such a fundamental aspect of manifolds. (See [BotT, Bred1, Mas, MilS, Spa], for instance.)

A more involved fact is that rational Pontrjagin classes can be defined for generalized manifolds. (See the introduction to [Bry+].)

For *smooth* manifolds, the definition of the Pontrjagin classes is classical. (See [BotT, MilS].) More precisely, one can define Pontrjagin classes for vector bundles in general, and then apply this to the tangent bundle of a smooth manifold to get the Pontrjagin classes of a manifold. As *integral* cohomol-

ogy classes, the Pontrjagin classes are preserved by diffeomorphisms between smooth manifolds, but not, in general, by homeomorphisms. However, a famous theorem of Novikov is that the Pontrjagin classes of smooth manifolds are preserved as *rational* cohomology classes by homeomorphisms in general. Further developments lead to the definition of rational characteristic classes on more general spaces.

For finite polyhedra, there is an earlier treatment of rational Pontrjagin classes, which goes back to work of Thom and Rohlin and Schwarz. See Section 20 of [MilS]. More precisely, this gives a procedure by which to define rational Pontrjagin classes for finite polyhedra which are generalized manifolds, and which is invariant under piecewise-linear equivalence. (For this, the generalized-manifold condition can be given in terms of rational coefficients for the homology groups.) If one starts with a smooth manifold, then there it can be converted to a piecewise-linear manifold (unique up to equivalence) by earlier results, and the classical rational Pontrjagin classes for the smooth manifold are the same as the ones that are obtained by the procedure for polyhedral spaces. (See [MilS] for more information.)

The results mentioned above indicate some of the ways that generalized manifolds are like ordinary manifolds. In particular, there is a large extent to which one can work with them, including making computations or measurements on them, in ways that are similar to those for ordinary manifolds.

This is pretty good! This is especially nice given the troubles that can come with homeomorphisms, as in Section 3 and Appendix C. I.e., homeomorphisms can be difficult to get or have, to begin with; even if they exist, they may necessarily be irregular, as in the case of double-suspensions of homology spheres [SieS], manifold factors (Subsection C.5), some other classical decomposition spaces (Subsections C.6, and C.7), and homeomorphisms between 4-dimensional manifolds (see [DonK, DonS, Fre, FreQ]); even if homeomorphisms exist and are of a good regularity class, their complexity may have to be very large, as in Section 3 and the results in [BooHP].

In the case of finite polyhedra, the condition of being a generalized manifold involves looking at the homology groups of the links (as mentioned before), and this is something which behaves in a fairly nice and stable way. Compare with Appendix E. By contrast, the property of being a manifold involves the fundamental groups of the links (at least in codimension 1 for having topological manifolds), and this is much more complicated. We have run into this already, in Section 3.

More generally, we have also seen in Appendix C how conditions related

to localized fundamental groups can arise, in connection with the existence of homeomorphisms and local coordinates. This includes the vanishing of π_1 in some punctured neighborhoods of points in topological manifolds of dimension at least 3. With generalized manifolds, one has certain types of special structure, but one does not necessarily have localized π_1 conditions like these. This gives a lot of simplicity and stableness, as well as flexibility.

E Some simple facts related to homology

Let P be a finite polyhedron, which we shall think of as being presented to us as a finite simplicial complex. Fix a positive integer k, and imagine that one is interested in knowing whether the homology $H_k(P)$ of P (with coefficients in $\mathbf{Z}$) is 0 in dimension k. This is a question that can be decided by an algorithm, unlike the vanishing of the fundamental group. This fact is standard and elementary, but not necessarily too familiar in all quarters. It came up in Section 3, and, for the sake of completeness, a proof will be described here. (Part of the point is to make it clear that there are no hidden surprises that are too complicated. We shall also try to keep the discussion elementary and direct, with a minimum of machinery involved.)

The first main point is that it suffices to consider *simplicial* homology of P, rather than something more general and elaborate (like *singular* homology). This puts strong limits on the type of objects with which one works. By contrast, note that the higher homotopy groups of a finite complex need not be finitely-generated. See Example 17 on p509 of [Spa]. This is true, but quite nontrivial, for simply-connected spaces. See Corollary 16 on p509 of [Spa].

Let us begin by considering the case of homology with coefficients in the rational numbers, rather than the integers. In this situation the homology is a vector space over $\mathbf{Q}$, and this permits the solution of the problem through means of linear algebra. To be explicit, one can think of the vanishing of the homology in dimension k in the following terms. One starts with the set of k-dimensional *chains* in P (with coefficients in $\mathbf{Q}$), which is the set of formal sums of oriented k-dimensional simplices with coefficients in $\mathbf{Q}$. This is a vector space, and multiplication by -1 is identified with reversing orientations on the simplices. The set of k-dimensional *cycles* then consists of k-dimensional chains with "boundary" equal to 0. This can be described by a finite set of linear equations in our vector space of k-dimensional chains.

The set of k-dimensional *boundaries* consist of k-dimensional chains which are themselves boundaries of $(k+1)$-dimensional chains. This is the same as taking the span of the boundaries of $(k+1)$-dimensional simplices in P, a finite set. Chains which are boundaries are automatically cycles, and the vanishing of the k-dimensional homology of P (with coefficients in $\mathbf{Q}$) is equivalent to the equality of the vector space of cycles with the vector space of boundaries. The problem of deciding whether this equality holds can be reduced to computations of determinants, for instance, and one can use other elementary techniques from linear algebra too.

In working with integer coefficients, one has the same basic definitions of chains, cycles, and boundaries, but now with coefficients in $\mathbf{Z}$ rather than $\mathbf{Q}$. The spaces of chains, cycles, and boundaries are now abelian groups, and the equations describing cycles and boundaries make sense in this context. "Abelian" is a key word here, and a crucial difference between this and the fundamental group. Homotopy groups π_j in dimensions $j \geq 2$ are always abelian too, but they do not come with such a simple presentation.

Let us be more explicit again. The space of k-dimensional chains in P, now with integer coefficients, is a free abelian group, generated by the k-dimensional simplices in P. One can think of it as being realized concretely by $\mathbf{Z}^r$, where r is the number of k-dimensional simplices in P.

The set of *cycles* among the chains is defined by a finite number of linear equations, as above. That is, the cycles $z \in \mathbf{Z}^r$ are determined by finitely many equations of the form

$$a_1 z_1 + a_2 z_2 + \cdots + a_r z_r = 0, \quad z = (z_1, z_2, \ldots, z_r) \in \mathbf{Z}^r, \tag{E.1}$$

where the a_i's are themselves integers. In fact, for the equations which actually arise in this situation, the a_i's are always either 0, 1, or -1. It will be useful, though, to allow for general vectors $(a_1, a_2, \ldots, a_r)$ in this discussion, and our computations will lead us to that anyway.

We begin with an observation about sets of vectors in $\mathbf{Z}^r$ defined by a single homogeneous linear equation (with integer coefficients).

Lemma E.2 *Let r be a positive integer, and let $a = (a_1, a_2, \ldots, a_r)$ be a vector of integers. Set*

$$C_a = \{z \in \mathbf{Z}^r : a_1 z_1 + a_2 z_2 + \cdots + a_r z_r = 0\}. \tag{E.3}$$

Then there is a group homomorphism $\phi_a : \mathbf{Z}^r \to \mathbf{Z}^r$ such that $\phi_a(\mathbf{Z}^r) = C_a$ and $\phi_a(z) = z$ when $z \in C_a$. This homomorphism can be effectively constructed given r and a.

Let us prove Lemma E.2. Let r and a be given, as above. We may as well assume that the a is not the zero vector, since if it were, C_a would be all of $\mathbf{Z}^r$, and we could take ϕ_a to be the identity mapping.

We may also assume that the components a_i of a are not all divisible by an integer different from 1 or -1, since we can always cancel out common factors from the a_i's without changing C_a.

Sublemma E.4 *Under these conditions (a is not the zero vector, and no integers divide all of the components of a except ± 1), there exists a vector $b = (b_1, b_2, \ldots, b_r)$ of integers such that*

$$a_1 b_1 + a_2 b_2 + \cdots + a_r b_r = 1. \tag{E.5}$$

A choice of b can be constructed effectively given a.

The $r = 2$ version of this is solved by the well known "Euclidean algorithm". It is more commonly formulated as saying that if one has two nonzero integers c and d, then one can find integers e and f such that $ce + df$ is equal to the (positive) greatest common divisor of c and d. This can be proved using the more elementary "division algorithm", to the effect that if m and n are positive integers, with $m < n$, then one can write n as $jm + i$, where j is a positive integer and $0 \le i < m$.

For general r one can reduce to the $r = 2$ case using induction, for instance. This is not hard to do, and we omit the details. (We should perhaps say also that the $r = 1$ case makes sense, and is immediate, i.e., a_1 has to be ± 1.)

Let us return now to Lemma E.2. Given any $z \in \mathbf{Z}^r$, write $z \cdot a$ for $z_1 c_1 + z_2 c_2 + \cdots + z_r c_r$. Define $\phi_a : \mathbf{Z}^r \to \mathbf{Z}^r$ by putting

$$\phi_a(z) = z - (z \cdot a)\, b, \tag{E.6}$$

where b is chosen as in Sublemma E.4. This is clearly a group homomorphism (with respect to addition). If z lies in C_a, then $z \cdot a = 0$, and $\phi_a(z) = z$. In particular, $\phi_a(\mathbf{Z}^r) \supseteq C_a$. On the other hand, if z is any element of $\mathbf{Z}^r$, then

$$\phi_a(z) \cdot a = z \cdot a - (z \cdot a)(b \cdot a), \tag{E.7}$$

and this is equal to 0, by (E.5). Thus $\phi_a(z) \in C_a$ for all $z \in \mathbf{Z}^r$, so that $\phi_a(\mathbf{Z}^r) \subseteq C_a$ too. This proves Lemma E.2.

Next we consider the situation of sets in $\mathbf{Z}^r$ defined by multiple linear equations (with linear coefficients).

Lemma E.8 *Let $a^1, a^2, \ldots, a^p$ be a collection of vectors in $\mathbf{Z}^r$, and define $C \subseteq \mathbf{Z}^r$ by*

$$C = \{z \in \mathbf{Z}^r : a^i \cdot z = 0 \text{ for } i = 1, 2, \ldots, p\}. \tag{E.9}$$

Then there is a group homomorphism $\psi : \mathbf{Z}^r \to \mathbf{Z}^r$ such that $\psi(\mathbf{Z}^r) = C$ and $\psi(z) = z$ when $z \in C$. This homomorphism can be effectively constructed given r and the vectors $a^1, a^2, \ldots, a^p$.

To prove this, we use induction, on p. The point is basically to iterate the construction of Lemma E.2, but one has to be a bit careful not to disrupt the previous work with the new additions.

When $p = 1$, Lemma E.8 is the same as Lemma E.2. Now suppose that we know Lemma E.8 for some value of p, and that we want to verify it for $p + 1$.

Let $a^1, a^2, \ldots, a^{p+1}$ be a collection of vectors in $\mathbf{Z}^r$, and let C^{p+1} denote the set of vectors $z \in \mathbf{Z}^r$ which satisfy $a^i \cdot z = 0$ for $i = 1, 2, \ldots, p + 1$, as in (E.9). Similarly, let C^p denote the set of $z \in \mathbf{Z}^r$ such that $a^i \cdot z = 0$ when $1 \le i \le p$. Our induction hypothesis implies that there is a group homomorphism $\psi^p : \mathbf{Z}^r \to \mathbf{Z}^r$ such that $\psi^p(\mathbf{Z}^r) = C^p$ and $\psi^p(z) = z$ when $z \in C^p$, and which can be effectively constructed given r and $a^1, a^2, \ldots, a^p$. We want to produce a similar homomorphism ψ^{p+1} for C^{p+1}.

The basic idea is to apply Lemma E.2 to the vector a^{p+1}. This does not quite work, and so we first modify a^{p+1} to get a vector which has practically the same effect as a^{p+1} for defining C^{p+1}, and which is in a more convenient form.

Specifically, let α be the vector in $\mathbf{Z}^r$ such that

$$\alpha \cdot z = a^{p+1} \cdot \psi^p(z) \qquad \text{for all } z \in \mathbf{Z}^r. \tag{E.10}$$

It is easy to compute α given a^{p+1} and ψ^p. One can also describe α as the vector which results by applying the *transpose* of ψ^p to a^{p+1}.

For our purposes, the main properties of α are as follows. First,

$$\alpha \cdot z = a^{p+1} \cdot z \qquad \text{when } z \in C^p. \tag{E.11}$$

This is a consequence of (E.10) and the fact that $\psi^p(z) = z$ when $z \in C^p$. Second,

$$\alpha \cdot \psi^p(z) = \alpha \cdot z \qquad \text{for all } z \in \mathbf{Z}^r. \tag{E.12}$$

To see this, note that $\psi^p(\psi^p(z)) = \psi^p(z)$ for all z, since ψ^p maps $\mathbf{Z}^r$ into C^p and ψ^p is equal to the identity on C^p, by construction. This and (E.10) give (E.12).

Because of (E.11), we have that

$$C^{p+1} = \{z \in C^p : \alpha \cdot z = 0\}. \tag{E.13}$$

That is, $C^{p+1} = \{z \in C^p : a^{p+1} \cdot z = 0\}$, by the definition of C^{p+1} and C^p, and then (E.13) follows from this and (E.11).

If $\alpha = 0$, then $C^{p+1} = C^p$, and we can stop here. That is, we can use ψ^p for ψ^{p+1}. Thus we assume instead that α is not the zero vector.

Let α' denote the vector in $\mathbf{Z}^r$ obtained from α by eliminating all common factors of the components of α which are positive integers greater than 1. In particular, α is a positive integer multiple of α'. From (E.12) and (E.13), we get that

$$\alpha' \cdot \psi^p(z) = \alpha' \cdot z \qquad \text{for all } z \in \mathbf{Z}^r \tag{E.14}$$

and

$$C^{p+1} = \{z \in C^p : \alpha' \cdot z = 0\}. \tag{E.15}$$

Now let us apply Sublemma E.4 to obtain a vector $\beta' \in \mathbf{Z}^r$ such that

$$\alpha' \cdot \beta' = 1. \tag{E.16}$$

Put $\beta'' = \psi^p(\beta')$. Then

$$\alpha' \cdot \beta'' = 1, \tag{E.17}$$

because of (E.14). Also, β'' lies in C^p, since ψ^p maps $\mathbf{Z}^r$ into C^p. (It was for this purpose that we have made the above modifications to a^{p+1}, i.e., to get $\beta'' \in C^p$.)

Now that we have all of this, we can proceed exactly as in the proof of Lemma E.2. Specifically, define $\phi' : \mathbf{Z}^r \to \mathbf{Z}^r$ by

$$\phi'(z) = z - (z \cdot \alpha')\, \beta''. \tag{E.18}$$

A key point is that

$$\phi'(C^p) \subseteq C^p, \tag{E.19}$$

which holds since $\beta'' \in C^p$, as mentioned above. Also,

$$\phi'(z) \cdot \alpha' = 0 \qquad \text{for all } z \in \mathbf{z}^r, \tag{E.20}$$

because of (E.17) and the definition (E.18) of $\phi'(z)$. From this and (E.19) it follows that

$$\phi'(C^p) \subseteq C^{p+1}, \tag{E.21}$$

using also (E.15).

On the other hand,

$$\phi'(z) = z \qquad \text{whenever } z \in \mathbf{Z}^r,\ z \cdot \alpha' = 0 \tag{E.22}$$

(by the definition (E.18) of ϕ'). In particular, $\phi'(z) = z$ when $z \in C^{p+1}$ (using (E.15) again). Combining this with (E.21), we get that

$$\phi'(C^p) = C^{p+1}. \tag{E.23}$$

Now we are almost finished with the proof. Define $\psi^{p+1} : \mathbf{Z}^r \to \mathbf{Z}^r$ by $\psi^{p+1} = \phi' \circ \psi^p$. This defines a group homomorphism (with respect to the usual addition of vectors), since ψ^p and ϕ' are group homomorphisms. We also have that

$$\psi^{p+1}(\mathbf{Z}^r) = C^{p+1}; \tag{E.24}$$

this follows from (E.23) and the fact that $\psi^p(\mathbf{Z}^r) = C^p$, which was part of our "induction hypothesis" on ψ^p.

Let us verify that

$$\psi^{p+1}(z) = z \qquad \text{whenever } z \in C^{p+1}. \tag{E.25}$$

If $z \in C^{p+1}$, then $z \in C^p$ in particular. This implies that $\psi^p(z) = z$, again by our "induction hypothesis" for ψ^p. Thus $\psi^{p+1}(z) = \phi'(z)$ in this case. We also have that $\phi'(z) = z$ if $z \in C^{p+1}$, because of (E.22) (and (E.15)). This gives (E.25), as desired.

It is easy to see from this construction that ψ^{p+1} is effectively computable from the knowledge of r and $a^1, a^2, \ldots, a^{p+1}$, given the corresponding assertion for ψ^p. Thus ψ^{p+1} has all the required properties. This completes the proof of Lemma E.8.

Now let us return to the original question, about determining whether the integral homology of a given finite complex vanishes in a given dimension. We can put this into a purely algebraic form, as follows. Suppose that a positive integer r is given, as well as two finite collections of vectors in $\mathbf{Z}^r$, $a^1, a^2, \ldots, a^p$, and $d^1, d^2, \ldots, d^q$. Given this data, the problem asks,

(E.26) is it true that for every $z \in \mathbf{Z}^r$ which satisfies $a^i \cdot z = 0$ for all $i = 1, 2, \ldots, p$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbf{Z}$ such that $z = \lambda_1 d^1 + \lambda_2 d^2 + \cdots + \lambda_q d^q$?

The question of vanishing of homology is of this form, and so an effective procedure for determining an answer of "yes" or "no" to (E.26) also provides a way to decide whether the homology of a given finite complex vanishes in a given dimension.

Let us use Lemma E.8 to reduce (E.26) to a simpler problem, as follows. Let $r \in \mathbf{Z}_+$ and $d^1, d^2, \ldots, d^q \in \mathbf{Z}^r$ be given, and also another vector $z \in \mathbf{Z}^r$. Given this data, the new problem asks,

(E.27) is it true that there exist $\lambda_1, \lambda_2, \ldots, \lambda_q \in \mathbf{Z}$ such that $z = \lambda_1 d^1 + \lambda_2 d^2 + \cdots + \lambda_q d^q$?

To show that (E.26) can be reduced to (E.27), let us first mention an auxiliary observation.

Suppose that vectors $a^1, a^2, \ldots, a^p$ in $\mathbf{Z}^r$ are given. Consider the set

$$C = \{w \in \mathbf{Z}^r : a^i \cdot w = 0 \text{ for } i = 1, 2, \ldots, p\}. \tag{E.28}$$

Then there is a finite collection of vectors $z^j \in C$ which generate C (as an abelian group), and which can be obtained through an effective procedure (given r and $a^1, a^2, \ldots, a^p$). This follows from Lemma E.8. Specifically, for the z^j's one can take $\psi^p(e^j)$, $1 \le j \le r$, where $\psi^p : \mathbf{Z}^r \to \mathbf{Z}^r$ is the homomorphism provided by Lemma E.8, and e^j is the jth standard basis vector in $\mathbf{Z}^r$, i.e., the vector whose jth component is 1 and whose other components are 0. From Lemma E.8, we know that ψ^p maps $\mathbf{Z}^r$ onto C, and that ψ^p can be effectively produced given r and $a^1, a^2, \ldots, a^p$. This implies that $z^j = \psi^p(e^j)$, $1 \le j \le r$, generate C and can be effectively produced as well. (One might analyze this further to reduce the number of vectors in the generating set, but this is not needed for the present purposes.)

Thus, in order to determine the answer to the question in (E.26), one can first apply this observation to produce a finite set of generators z^j for C as in (E.28). An answer of "yes" for the question in (E.26) is then equivalent to having an answer of "yes" for the question in (E.27) for each z^j (in the role of z in (E.27)). In this way, we see that an algorithm for deciding the answer to (E.27) gives rise to an algorithm for determining the answer to (E.26).

Now let us consider (E.27). Imagine that $r, q \in \mathbf{Z}^r$ and $d^\ell \in \mathbf{Z}^r$, $1 \le \ell \le q$, are given. We want to know if there is a vector $\lambda \in \mathbf{Z}^q$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$, such that

$$z = \lambda_1 d^1 + \lambda_2 d^2 + \cdots + \lambda_q d^q. \tag{E.29}$$

Let us rewrite this as

$$z_i = \lambda_1 d_i^1 + \lambda_2 d_i^2 + \cdots + \lambda_q d_i^q \qquad \text{for } i = 1, 2, \ldots, r, \tag{E.30}$$

where z_i, d_i^ℓ, $1 \le i \le r$, denote the ith components of z, d^ℓ, respectively.

Define vectors $\delta^i \in \mathbf{Z}^q$ by $\delta_j^i = d_i^j$ for $j = 1, 2, \ldots, q$. With this use of "transpose" we can rewrite (E.30) as

$$z_i = \lambda \cdot \delta^i \qquad \text{for } i = 1, 2, \ldots, r. \tag{E.31}$$

Here "$\cdot$" denotes the usual dot product for vectors, although now for vectors in $\mathbf{Z}^q$, rather than $\mathbf{Z}^r$, as before.

With (E.31), we are in a somewhat similar situation as we have considered before, but with inhomogeneous equations rather than homogeneous ones. Note that for the inhomogeneous equations there can be issues of torsion, i.e., it may be that no solution $\lambda \in \mathbf{Z}^q$ to (E.31) exists, but a solution does exist if one replaces z_i with nz_i for some positive integer n.

We want to have an effective procedure for deciding when a vector $\lambda \in \mathbf{Z}^q$ exists which provides a solution to (E.31). To do this, we shall try to systematically reduce the number of equations involved. Consider first the equation with $i = 1$, i.e.,

$$z_1 = \lambda \cdot \delta^1. \tag{E.32}$$

If there is no $\lambda \in \mathbf{Z}^q$ which satisfies this single equation, then there is no solution for the system (E.31) either. If there are solutions to this equation, then we can try to analyze the remaining equations on the set of λ's which satisfy this equation.

In fact it is easy to say exactly when there is a $\lambda \in \mathbf{Z}^q$ which satisfies (E.32). A necessary condition is that z_1 be divisible by all nonzero integers that divide each component δ_j^1, $1 \le j \le q$, of δ^1. (If z_1 or some δ_j^1 is 0, then it is divisible by all integers.) This necessary condition is also sufficient, because of Sublemma E.4. The validity or not of this condition can be determined effectively, and, when the condition holds, a particular solution $\widetilde{\lambda}$ of (E.32) can be produced effectively from the knowledge of z_1 and δ^1, because of Sublemma E.4.

If no solution to (E.32) exists in $\mathbf{Z}^q$, then one can simply stop, as the answer to the question of the existence of a solution to the system (E.31) is then known to be "no". Let us suppose therefore that there is at least one solution to (E.32). As in the preceding paragraph, this means that there is a solution $\widetilde{\lambda}$ which can be effectively computed from the data.

Set $L_1 = \{\lambda \in \mathbf{Z}^q : z_1 = \lambda \cdot \delta^1\}$. The existence of a solution $\lambda \in \mathbf{Z}^q$ to the original system of equations in (E.31) is equivalent to the existence of a $\lambda \in L_1$ which satisfies

$$z_i = \lambda \cdot \delta^i \qquad \text{for } i = 2, \ldots, r. \tag{E.33}$$

Thus we have reduced the number of equations involved, at the cost of having a possibly more complicated set of λ's as admissible competitors.

However, we can rewrite L_1 as

$$L_1 = \{\lambda \in \mathbf{Z}^q : (\lambda - \widetilde{\lambda}) \cdot \delta^1 = 0\}. \tag{E.34}$$

Set $L_1' = \{\tau \in \mathbf{Z}^r : \tau \cdot \delta^1 = 0\}$. We can reformulate the question of whether there is a $\lambda \in L_1$ such that (E.33) holds as asking whether there is a $\tau \in L_1'$ such that

$$z_i + \widetilde{\lambda} \cdot \delta^i = \tau \cdot \delta^i \qquad \text{for } i = 2, \ldots, r. \tag{E.35}$$

In other words, we can make a change of variables and modify the equations slightly so that the set L_1' in which we look for solutions is defined by a *homogeneous* equation. This permits us to apply Lemma E.2 (with r replaced by q) to get a homomorphism $\phi : \mathbf{Z}^q \to \mathbf{Z}^q$ such that $\phi(\mathbf{Z}^q) = L_1'$. Using this, our question now becomes the following:

$$\begin{aligned} &\text{does there exist } \xi \in \mathbf{Z}^q \text{ such that} \\ &z_i + \widetilde{\lambda} \cdot \delta^i = \phi(\xi) \cdot \delta^i \text{ for } i = 2, \ldots, r? \end{aligned} \tag{E.36}$$

This is equivalent to the earlier question, because the set of vectors $\tau \in \mathbf{Z}^q$ which lie in L_1' is the same as the set of vectors of the form $\phi(\xi)$, where ξ is allowed to be any element of $\mathbf{Z}^q$.

From Lemma E.2 we know that ϕ can be effectively computed from the knowledge of q and δ^1. By making straightforward substitutions, we can rewrite the equations in (E.36) as

$$\widehat{z}_i = \xi \cdot \widehat{\delta}^i \qquad \text{for } i = 2, \ldots, r, \tag{E.37}$$

where the integers $\widehat{z}_i$ and vectors $\widehat{\delta}^i$ can be computed in terms of the original z_i's and δ^i's, $\widetilde{\lambda}$, and ϕ. In particular, they can be computed effectively in terms of the original data in the problem.

Thus we are back to the same kind of problem as we started with, asking about the existence of a vector in $\mathbf{Z}^q$ which satisfies a family of inhomogeneous linear equations. However, now we have reduced the number of equations by 1. By repeating the process, we can reduce to the case of a single inhomogeneous equation, which we know how to solve (as we saw before).

This shows that there is an effective method to determine the answer to the question in (E.27). As indicated earlier, this also gives a method for deciding the answer to the question in (E.26), and to the original problem about vanishing of homology in a finite complex.

Let us briefly mention a cruder and more naive approach to (E.27). If the answer to (E.27) is "yes", then it means that there do exist integers $\lambda_1, \lambda_2, \ldots, \lambda_q$ which satisfy $z = \lambda_1 d^1 + \cdots + \lambda_q d^q$. To look for an answer of "yes" to (E.27), one can simply start searching among all vectors $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)$ in $\mathbf{Z}^q$, stopping one finds a λ which satisfies the equation above.

If the answer to the question in (E.27) is "no", then this search will not produce an answer in a finite amount of time. However, in this case one can make a "dual" search to find a reason for the vector z not to be in the subgroup of $\mathbf{Z}^r$ generated by $d^1, d^2, \ldots, d^q$, a reason which can also be found in finite time, when it exists. Specifically, z does not lie in the subgroup of $\mathbf{Z}^r$ generated by $d^1, d^2, \ldots, d^q$ if and only if there is a homomorphism $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ such that $\sigma(d^j) = 0$ for each j but $\sigma(z) \neq 0$. We shall explain why this is true in a moment, but first let us notice how this "reason" for an answer of "no" does fit our purpose.

A homomorphism $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ can be described by r elements of $\mathbf{Q}/\mathbf{Z}$, i.e., the values of σ on the r standard basis vectors in $\mathbf{Z}^r$. Any elements of $\mathbf{Q}/\mathbf{Z}$ can be used here, and elements of $\mathbf{Q}/\mathbf{Z}$ can be described in finite terms, i.e., by pairs of integers. The conditions $\sigma(d^j) = 0$, $1 \le j \le q$, and $\sigma(z) \neq 0$, can be verified in finite time in a straightforward manner. Thus, if a $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ exists with these properties, then it can be found in a finite amount of time, through an exhaustive search.

If no such σ exists, then this exhaustive search will not stop in finite time. However, the exhaustive search for the λ's will stop in a finite time in this case. Thus one can run the two searches in parallel, and stop whenever one of them stops. One of the two searches will always stop in a finite amount of time, and thereby give an answer of "yes" or "no" to the original question (about whether z lies in the subgroup of $\mathbf{Z}^r$ generated by $d^1, d^2, \ldots, d^q$).

Although naive, this argument fits nicely with what happens for the prob-

lem of deciding whether the fundamental group of a finite complex is trivial. When the answer is "yes", one can find this out in finite time, again through exhaustive searches. In algebraic terms, one searches for realizations of the generators of the fundamental group as trivial words, using the relations for the fundamental group which can be read off from the given complex. In general there is no finite test for the nontriviality of words, however, and indeed the original question is not algorithmically decidable.

In some cases, one might have extra information which does allow for effective tests for answers of "no", and abelian groups are a very special instance of this.

Let us come back now to the assertion above, that $z \in \mathbf{Z}^r$ does not lie in the subgroup generated by $d^1, d^2, \ldots, d^q \in \mathbf{Z}^r$ if and only if there is a homomorphism $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ such that $\sigma(d^j) = 0$ for $j = 1, 2, \ldots, q$ and $\sigma(z) \neq 0$. Of course this is closely analogous to familiar statements about vectors in a vector space and linear mappings into the ground field.

The "if" part of the statement above is immediate, and so it suffices to consider the "only if" part. Thus we assume that z does not lie in the subgroup generated by the d^j's, and we want to find a homomorphism $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ with the required properties.

We begin by setting σ to be 0 on the subgroup generated by $d^1, d^2, \ldots, d^q$. For $\sigma(z)$ we have to be slightly careful. If there is a positive integer n such that nz lies in the subgroup generated by $d^1, d^2, \ldots, d^q$, then we need to choose $\sigma(z)$ so that $n\sigma(z) = 0$. If no such n exists, take $\sigma(z)$ to be the element of $\mathbf{Q}/\mathbf{Z}$ corresponding to $1/2 \in \mathbf{Q}$. If such an n does exist, let n_0 be the smallest positive integer with that property. Thus $n_0 > 1$, since z itself does not lie in the subgroup generated by the d^j's. In this case we take $\sigma(z)$ to be the element of $\mathbf{Q}/\mathbf{Z}$ which corresponds to $1/n_0$. This element is not 0, since $n_0 > 1$, but $\sigma(n_0 z)$ is then 0 in $\mathbf{Q}/\mathbf{Z}$.

We now extend σ to the subgroup generated by z and the d^j's, in the obvious way (so that σ is a homomorphism). One should be a bit careful here too, i.e., that this can be done in a consistent manner, so that σ really is well-defined on the subgroup generated by z and the d^j's. This comes down to the fact that if n is an integer such that nz lies in the subgroup generated by $d^1, d^2, \ldots, d^q$, then n should be divisible by n_0, which ensures that $\sigma(nz)$ is equal to 0 in $\mathbf{Q}/\mathbf{Z}$. These things are not hard to check.

Now we simply want to extend σ to all of $\mathbf{Z}^r$, in such a way that it is still a homomorphism into $\mathbf{Q}/\mathbf{Z}$. This is not difficult to do; for a general assertion along these lines, see Theorem 4.2 on p312 of [Mas]. The main

point is that $\mathbf{Q}/\mathbf{Z}$ is *divisible*, which means that for each element x of $\mathbf{Q}/\mathbf{Z}$ and each nonzero integer m, there is a $y \in \mathbf{Q}/\mathbf{Z}$ such that $my = x$. In order to extend σ to all of $\mathbf{Z}^r$, one can extend it to new elements one at a time, and to the subgroups that they generate together with the subgroup of $\mathbf{Z}^r$ on which σ is already defined. The divisibility property of $\mathbf{Q}/\mathbf{Z}$ guarantees that there are always values available in $\mathbf{Q}/\mathbf{Z}$ by which to make well-defined extensions. That is, if σ is already defined on some subgroup H of $\mathbf{Z}^r$, and w is an element in $\mathbf{Z}^r$ not in H, then there is always a point in $\mathbf{Q}/\mathbf{Z}$ to use as the value of σ at w. This is not a problem if nw does not lie in H for any nonzero integer n — in which case one could just as well take $\sigma(w) = 0$ — but if $nw \in H$ for some nonzero n, then one has to choose $\sigma(w)$ so that $n\sigma(w)$ is equal to $\sigma(nw)$, where the latter is already been determined by the definition of σ on H.

By repeating this process, one can eventually extend σ so that it becomes a homomorphism from all of $\mathbf{Z}^r$ into $\mathbf{Q}/\mathbf{Z}$. For instance, one can apply this process to the standard basis vectors in $\mathbf{Z}^r$, at least when they are not already included in the subgroup of $\mathbf{Z}^r$ on which σ has already been defined (at the given stage of the construction). In the end one obtains a homomorphism $\sigma : \mathbf{Z}^r \to \mathbf{Q}/\mathbf{Z}$ such that $\sigma(d^j) = 0$ for $1 \le j \le q$ and $\sigma(z) \neq 0$, as desired.

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